MATH 321:
Lecture 6

PROBABILITY BASICS (§2.1-2.2)

Defs

- $\Omega = \{H,T\}^N = \{\omega = (\omega_1, \dots, \omega_N) \mid \omega_i \in \{H,T\} \ \forall i \in \{1,\dots,N\}\}$ is called coin-toss space.

- A probability measure $\mathbb{Q}$ is called a binomial product measure if $\exists \ \alpha, \beta \in (0,1)$ s.t.

$$\alpha + \beta = 1$$

$$\& \quad \mathbb{Q}(\omega) = \alpha^{\#H(\omega)} \beta^{\#T(\omega)} \quad \forall \omega \in \Omega$$

where $\#H(\omega) = |\{\omega_i \in \omega \mid \omega_i = H\}|$ (the number of heads in ω) and $\#T(\omega) = |\{\omega_i \in \omega \mid \omega_i = T\}|$ (the number of tails in ω).

- A finite probability space $(\Omega, \mathbb{P})$ consists of a sample space Ω (a nonempty, finite set) and probability measure $\mathbb{P}$ (a function $\mathbb{P}: \Omega \to [0,1]$ s.t. $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$).

- An event is a subset $A \subseteq \Omega$, and $\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega)$.
  (Recall: ω is called an outcome. So events are sets of outcomes.)

Recall: Binomial Theorem: For any $x, y \in \mathbb{R}$ and any integer n,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Proposition $(\Omega, \mathbb{Q})$ is a finite probability space.

Proof. $\Omega = \{H,T\}^N$ is finite and nonempty.

Just check $\sum_{\omega} \mathbb{Q}(\omega) = 1$:

$$\sum_{\omega\in\Omega} \mathbb{Q}(\omega) = \sum_{\omega\in\Omega} \alpha^{\#H(\omega)} \beta^{\#T(\omega)} =$$

There are $\binom{N}{k}$ choices of which k flips were heads, for each k b/w 0 & N.

$$= \sum_{k=0}^{N} \binom{N}{k} \alpha^k \beta^{N-k} =$$

This is called the Binomial Theorem. Remember from Bridge?

$$= (\alpha+\beta)^N = 1^N = 1 \quad \blacksquare$$

Because $\alpha+\beta=1$

Proposition The pricing/risk-neutral measure $\tilde{\mathbb{P}}$ is a binomial product measure w/ $\alpha=\tilde{p}, \beta=\tilde{q}$.

Remark The reference probability measure need not be a binomial product measure! Just needs $\mathbb{P}(\omega)>0 \ \forall\, \omega\in\Omega$. (But later we'll need it to be. More later.)

Remark Binomial product measure $\longleftrightarrow$ random experiments with independent tosses of independent coins.

<u>Example</u>. Let $A_n = \{\omega \in \Omega : \omega_n = H\}$ ← n^{th} toss is H

$B_n = \{\omega \in \Omega : \omega_n = T\}$ ← n^{th} toss is T

Then for a binomial product measure $\mathbb{Q}$, find $\mathbb{Q}[A_N]$ and $\mathbb{Q}[B_N]$.

<u>Solution</u>: (We'll do $\mathbb{Q}[A_N]$ together – $\mathbb{Q}[B_N]$ is left as an exercise.)

$$\mathbb{Q}[A_N] = \sum_{\omega \in A_N} \mathbb{Q}(\omega) = \sum_{\omega \in A_N} \alpha^{\#H(\omega)} \beta^{\#T(\omega)} =$$

starts at 1 b/c we know there's at least 1H.

$$= \sum_{j=1}^{N} \binom{N-1}{j-1} \alpha^{j} \beta^{N-j}$$

out of the N-1 first tosses, which j-1 are H's?

change index using $i = j-1$. (so $j = i+1$)

$$= \sum_{i=0}^{N-1} \binom{N-1}{i} \alpha^{i+1} \beta^{N-i-1}$$

$$= \alpha \sum_{i=0}^{N-1} \binom{N-1}{i} \alpha^{i} \beta^{N-i-1}$$

Binomial Theorem

$$= \alpha(\alpha+\beta)^{N-1} = \alpha \quad \blacksquare$$

Defs

- A random variable X is any function $X : \Omega \to \mathbb{R}$.
- The expected value of X is $\mathbb{E}[X] = \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega)$ (Other notations: $\mathbb{E}(X)$, $\mathbb{E}X$.)
- The distribution of X is a specification of the probabilities that a random variable (R.V.) takes given variables

  ⇝ see discussion in §2.2 for an example that details why a R.V. and its distribution are not the same (change of measure).

Proposition (Linearity of Expectation)

For all R.V.'s X and Y and for all $a, b \in \mathbb{R}$,

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

Remarks

- $\tilde{\mathbb{E}}[X]$ will mean risk-neutral exp. val.
- If $g : \mathbb{R} \to \mathbb{R}$ is a function, X a R.V., then $g(X)$ is a R.V. and

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} \mathbb{P}(\omega)\, g(X(\omega)).$$

Defs • The variance of X is

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

• The standard deviation of X is

$$Sd(X) = \sqrt{Var(X)}.$$

Proposition (Non-linearity of variance)

$\forall\, a, b \in \mathbb{R}$, R.V. X

$$Var(aX + b) = a^2\, Var(X).$$

Def: • Covariance (measure of how two RV's vary together). Let X, Y be R.V.'s.

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

• X, Y are positively correlated if $Cov(X,Y) > 0$
• X, Y are negatively correlated if $Cov(X,Y) < 0$
• X, Y are uncorrelated if $Cov(X,Y) = 0$
• The correlation of X & Y is $\rho(X,Y) = \dfrac{Cov(X,Y)}{Sd(X)\,Sd(Y)}$
• X, Y are independent if

$$\mathbb{E}[f(X)\,g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]$$

$\forall$ functions $f : R(X) \to \mathbb{R}$ $\quad g : R(Y) \to \mathbb{R}$

"Range of X" "range of Y"

Corollary: • Independent implies uncorrelated.

• Uncorrelated DOES NOT imply independent (because covariance measures only linear dependence).

Example Find the variance of $X+Y$ for

(a) general R.V.'s X & Y

(b) independent R.V.'s X&Y.

Solution: (a) $\text{Var}(X+Y) = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$

$$= \mathbb{E}[X^2 + 2XY + Y^2] - \left(\mathbb{E}[X] + \mathbb{E}[Y]\right)^2$$

$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \left(\mathbb{E}[X]^2 + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y]^2\right)$$

$$= \underbrace{\mathbb{E}[X^2] - \mathbb{E}[X]^2}_{\text{Var}(X)} + \underbrace{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2}_{\text{Var}(Y)} + 2\underbrace{\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)}_{\text{Cov}(X,Y)}$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y).$$

(b) If X, Y are independent, $\text{Cov}(X,Y) = 0$.

So independence implies linearity of variance. ■

Proposition X and Y are independent $\iff \forall\, a, b$,

$$\mathbb{P}\Big(\underbrace{\{\omega\in\Omega \mid X(\omega) = a\}}_{\text{call this event } A} \cap \underbrace{\{\omega\in\Omega \mid Y(\omega) = b\}}_{\text{call this event } B}\Big) = \mathbb{P}(X=a)\,\mathbb{P}(Y=b)$$

i.e. $\mathbb{P}(A\cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.