MATH 321: LECTURE 7

2/10

From last time:

$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X]^2)]$, $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

Rmk: This is equivalent to

$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X^2]$.

Pf: Easy exercise! $\blacksquare$

Rmk: $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.

Example: uncorrelated but dependent RV's: Let X have the following "uniform distribution" on the set $\{-2,-1,0,1,2\}$. (i.e. each occurs w/ probability $\frac{1}{5}$). and let $Y = X^2$.

$$\Rightarrow \quad X^2 = \begin{cases} 4 & \text{w/ prob. } 2/5 \\ 1 & \text{w/ prob. } 2/5 \\ 0 & \text{w/ prob. } 1/5 \end{cases}$$

$$\mathbb{E}[X^2] = \frac{2}{5}(4) + \frac{2}{5}(1) + \frac{1}{5}(0) = 2$$

$$\mathbb{E}[X^3] = \mathbb{E}[X] = 0.$$

$$\Rightarrow \quad X^4 = \begin{cases} 16 & \text{w/ prob. } 2/5 \\ 1 & \text{w/ prob. } 2/5 \\ 0 & \text{w/ prob. } 1/5 \end{cases}$$

$$\mathbb{E}[X^4] = \frac{2}{5}(16) + \frac{2}{5}(1) + \frac{1}{5}(0) = \frac{34}{5} = 6.8.$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$
$$= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]$$
$$= 0 - 0 \cdot 2 = 0.$$

$\Rightarrow$ uncorrelated.

But consider function $f(X) = X^2$:

$$\mathbb{E}[f(X)Y] = \mathbb{E}[X^4] = 6.8 \neq 4 = \mathbb{E}[X^2]\mathbb{E}[X^2],$$

$\Rightarrow$ depedendent. ∎

Note: good exam prep!

Exercise: Suppose X & Y are independent RV's. Let $Z = X+Y$, $W = XY$. Are these dependent or independent RV's? (Provide a proof.)

→ See OHP problems to check your logic.

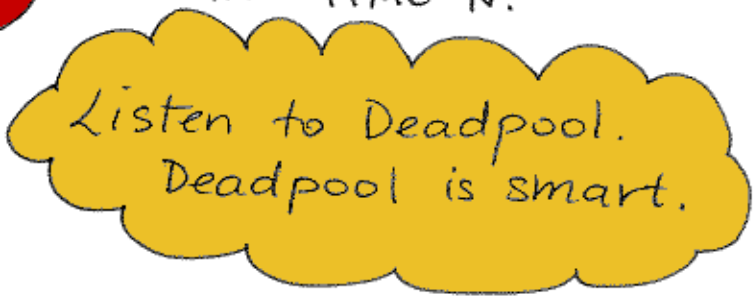
[Interlude on Convexity Goes Here]

§ 2.3 Conditional Expectation

Let $\mathbb{E}^{\mathbb{P}}$ refer to expectation in the finite probability space $(\Omega, \mathbb{P})$, and let X be a random variable on $(\Omega, \mathbb{P})$. Then we define $\mathbb{E}_n^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}}[X|A] =$

$$= \frac{1}{\mathbb{P}(A)} \sum_{\omega \in A} X(\omega)\mathbb{P}(\omega).$$

WHAT MY OLD-FASHIONED PAL IS TRYING TO SAY IS THAT $\mathbb{E}_n^{\mathbb{P}}[X]$ IS THE EXPECTED VALUE OF X BASED ON THE INFO AVAILABLE AT TIME N.



Fundamental Properties of Conditional Expectation

Let X, Y be R.V.'s, let $N \in \mathbb{Z}^+$, let $0 \leq n \leq N$.

(1) Linearity: $\forall c_1, c_2 \in \mathbb{R}$, $\mathbb{E}_n[c_1 X + c_2 Y]$

$$= c_1 \mathbb{E}_n[X] + c_2 \mathbb{E}_n[Y].$$

(2) Taking out what's known: If X only depends on the first n tosses, $\mathbb{E}_n[XY] = X\,\mathbb{E}_n[Y]$.

(3) Iterated conditioning: If $0 \leq n < m \leq N$, then

$$\mathbb{E}_n\big[\mathbb{E}_m[X]\big] = \mathbb{E}_n[X].$$

(4) Independence: If X depends only on tosses $n+1$ through N, then

$$\mathbb{E}_n[X] = \mathbb{E}[X].$$

(5) Conditional Jensen's Inequality: If φ is convex,

$$\mathbb{E}_n[\varphi(X)] \geq \varphi(\mathbb{E}_n[X]).$$

Notation

From backward induction, we know

$$S_n(\omega_1,\ldots,\omega_n) = \frac{1}{1+r}\left[\tilde{p}\,S_{n+1}(\omega_1,\ldots,\omega_n,H) + \tilde{q}\,S_{n+1}(\omega_1,\ldots,\omega_n,T)\right]$$

$$= \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}](\omega_1,\ldots,\omega_n) \qquad (\dagger)$$

So we can write

$$S_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}] \qquad (*)$$

Since $(\dagger)$ is true for all $(\omega_1,\ldots,\omega_n) \in \{H,T\}^n$.

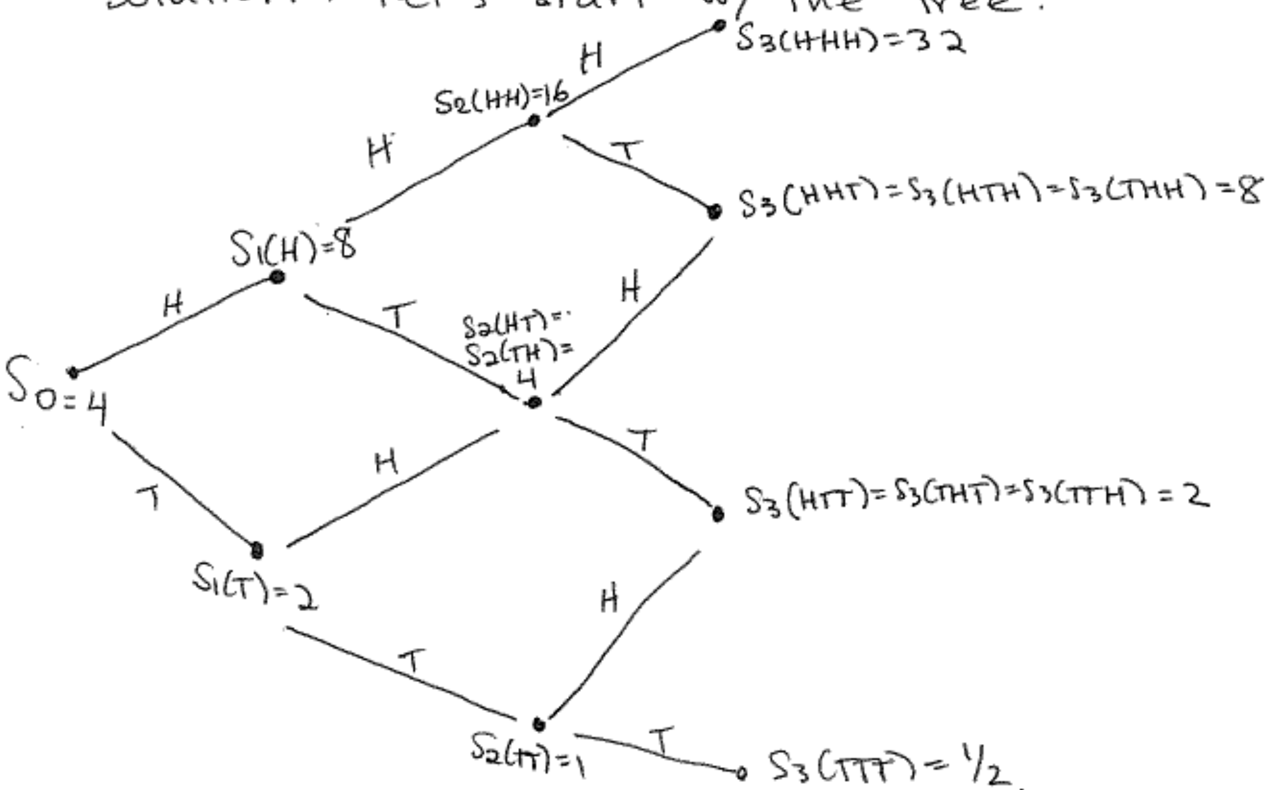
Example

Suppose $p = 2/3$, $q = 1/3$, $S_0 = 4$, $r = 5/4$, $u = 2$, $d = 1/2$

Compute $\mathbb{E}_1[S_2](H)$, $\mathbb{E}_1[S_3](H)$ & $\mathbb{E}_1[\mathbb{E}_2[S_3]](H)$.

(Note: $\mathbb{E}$ here represents the expected value under the reference probability measure. We're not pricing anything here.)

Solution: let's start w/ the tree:



So then

(i) $\mathbb{E}_1[S_2](H) = \frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 4 = 12$

(ii) $\mathbb{E}_1[S_3](H) = \frac{2}{3}\left[\frac{2}{3} \cdot 32 + \frac{1}{3} \cdot 8\right] + \frac{1}{3}\left[\frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2\right] = 18$

(iii) $\mathbb{E}_1[\mathbb{E}_2[S_3]](H) = \mathbb{E}_1[S_3](H)$ by iterated conditioning.

But let's compute it directly:

$$\mathbb{E}_2[S_3](HH) = \frac{2}{3} \cdot 32 + \frac{1}{3} \cdot 8 = 24$$

$$\mathbb{E}_2[S_3](HT) = \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2 = 6$$

and

$$\mathbb{E}_1[\mathbb{E}_2[S_3]](H) = \frac{2}{3}\mathbb{E}_2[S_3](HH) + \frac{1}{3}\mathbb{E}_2[S_3](HT)$$

$$= \frac{2}{3} \cdot 24 + \frac{1}{3} \cdot 6 = 18,$$

as expected. $\blacksquare$

Interpreting Equation (*) intuitively:

$$S_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}] = \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{1+r}\right]$$

I.E. OUR BEST GUESS FOR THE STOCK PRICE AT TIME $n+1$ (APPRORIATELY DISCOUNTED) IS JUST THE STOCK PRICE AT TIME n.

We have a name for processes like this:

- Processes $\{M_0, M_1, \ldots, M_N\}$ (sequence of RV's) where best guess for value of M_{n+1} at time n is just M_n are called martingales.

$\rightsquigarrow$ §2.4 in Shreve