MATH 321:

Lecture 8–9

§ 2.3 Conditional Expectation

Let $\mathbb{E}^{\mathbb{P}}$ refer to expectation in the finite probability space $(\Omega, \mathbb{P})$, and let X be a random variable on $(\Omega, \mathbb{P})$. Then we define $\mathbb{E}_n^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}}[X|A] = \frac{1}{\mathbb{P}(A)} \sum_{\omega \in A} X(\omega)\mathbb{P}(\omega)$.

WHAT MY OLD-FASHIONED PAL IS TRYING TO SAY IS THAT $\mathbb{E}_n^{\mathbb{P}}[X]$ IS THE EXPECTED VALUE OF X BASED ON THE INFO AVAILABLE AT TIME N.

Listen to Deadpool. Deadpool is smart.

Fundamental Properties of Conditional Expectation

Let X, Y be R.V.'s, let $N \in \mathbb{Z}^+$, let $0 \leq n \leq N$.

(1) Linearity: $\forall c_1, c_2 \in \mathbb{R}$, $\mathbb{E}_n[c_1 X + c_2 Y] = c_1 \mathbb{E}_n[X] + c_2 \mathbb{E}_n[Y]$.

(2) Taking out what's known: If X only depends on the first n tosses, $\mathbb{E}_n[XY] = X\,\mathbb{E}_n[Y]$.

(3) Iterated conditioning: If $0 \leq n < m \leq N$, then $\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X]$.

(4) Independence: If X depends only on tosses $n+1$ through N, then $\mathbb{E}_n[X] = \mathbb{E}[X]$.

(5) Conditional Jensen's Inequality: If φ is convex, $\mathbb{E}_n[\varphi(X)] \geq \varphi(\mathbb{E}_n[X])$.

Notation

From backward induction, we know

$$S_n(\omega_1,\dots,\omega_n) = \frac{1}{1+r}\left[\tilde{p}\, S_{n+1}(\omega_1,\dots,\omega_n,H) + \tilde{q}\, S_{n+1}(\omega_1,\dots,\omega_n,T)\right]$$
$$= \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}](\omega_1,\dots,\omega_n) \qquad (\dagger)$$

So we can write

$$S_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}] \qquad (*)$$

Since $(\dagger)$ is true for all $(\omega_1,\dots,\omega_n) \in \{H,T\}^n$.

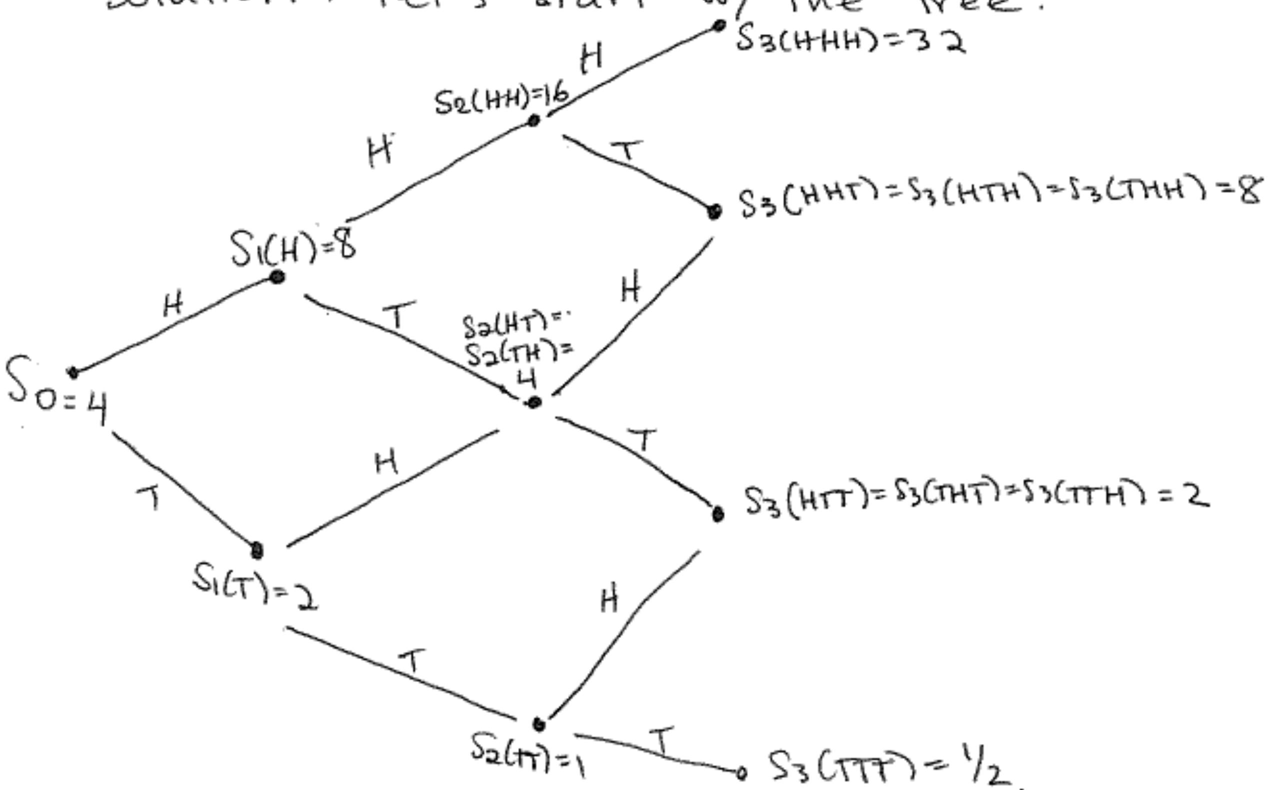
Example

Suppose $p = 2/3$, $q = 1/3$, $S_0 = 4$, $r = 5/4$, $u = 2$, $d = 1/2$

compute $\mathbb{E}_1[S_2](H)$, $\mathbb{E}_1[S_3](H)$ & $\mathbb{E}_1[\mathbb{E}_2[S_3]](H)$.

(Note: $\mathbb{E}$ here represents the expected value under the reference probability measure. We're not pricing anything here.)

Solution: let's start w/ the tree:



So then

(i) $\mathbb{E}_1[S_2](H) = \frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 4 = 12$

(ii) $\mathbb{E}_1[S_3](H) = \frac{2}{3}\left[\frac{2}{3} \cdot 32 + \frac{1}{3} \cdot 8\right] + \frac{1}{3}\left[\frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2\right] = 18$

(iii) $\mathbb{E}_1[\mathbb{E}_2[S_3]](H) = \mathbb{E}_1[S_3](H)$ by iterated conditioning.

But let's compute it directly:

$$\mathbb{E}_2[S_3](HH) = \frac{2}{3} \cdot 32 + \frac{1}{3} \cdot 8 = 24$$

$$\mathbb{E}_2[S_3](HT) = \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2 = 6$$

and

$$\mathbb{E}_1[\mathbb{E}_2[S_3]](H) = \frac{2}{3}\mathbb{E}_2[S_3](HH) + \frac{1}{3}\mathbb{E}_2[S_3](HT)$$

$$= \frac{2}{3} \cdot 24 + \frac{1}{3} \cdot 6 = 18,$$

as expected. ∎

Interpreting Equation (*) intuitively:

$$S_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[S_{n+1}] = \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{1+r}\right]$$

I.E. OUR BEST GUESS FOR THE STOCK PRICE AT TIME $n+1$ (APPRORIATELY DISCOUNTED) IS JUST THE STOCK PRICE AT TIME n.

We have a name for processes like this:

- Processes $\{M_0, M_1, \ldots, M_N\}$ (sequence of RV's) where best guess for value of M_{n+1} at time n is just M_n are called martingales.

$\rightsquigarrow$ §2.4 in Shreve

Def. A sequence of random variables $(M_n)_{0 \le n \le N}$ ("stochastic process") such that $\forall n \in \{0, 1, \ldots, N\}$, M_n depends only on the first n tosses (aka "adapted process") is a

- martingale $\Longleftrightarrow M_n = \mathbb{E}_n[M_{n+1}] \ \forall n$
- submartingale $\Longleftrightarrow M_n \le \mathbb{E}_n[M_{n+1}] \ \forall n$
- Supermartingale $\Longleftrightarrow M_n \ge \mathbb{E}_n[M_{n+1}] \ \forall n$

(This is called the "one step ahead property"!)

Remark: Note that the choice of measure matters! An adapted process can be a martingale under one measure but not another!

Proposition Let $(M_n)_{0 \le n \le N}$ be a martingale. Then

$$M_n = \mathbb{E}_n[M_m] \quad \forall \ 0 \le n \le m \le N$$

Proof Induction! Base case:

$$M_n = \mathbb{E}_n[M_{n+1}] = \mathbb{E}_n\left[\mathbb{E}_{n+1}[M_{n+2}]\right] = \mathbb{E}_n[M_{n+2}]$$

because it's a martingale

iterated conditioning

Induction step:

Assume $M_n = \mathbb{E}_n[M_{n+k}]$ for some k.

Then $M_n = \mathbb{E}_n[\mathbb{E}_{n+k}[M_{n+k+1}]] = \mathbb{E}_n[M_{n+k+1}]$.

So by induction, we can go as far ahead as we want. ∎

Corollary $M_0 = \mathbb{E}[M_N]$.

Claim. The adapted process

$$\left(\frac{S_n}{(1+r)^n}\right)_{0\le n\le N}$$

is a martingale under $\tilde{\mathbb{P}}$ (the risk-neutral measure).

Proof 1. We need to show $\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \frac{S_n}{(1+r)^n}$ for all $n \in \{0,1,\dots,N\}$:

Let $n \in \{0,1,\dots,N\}$.

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n[S_{n+1}]$$

taking out what's known

only 2 choices

$$= \frac{1}{(1+r)^{n+1}}\left\{\tilde{p}\,u S_n + \tilde{q}\,d S_n\right\}$$

$$= \frac{S_n}{(1+r)^{n+1}}\left\{\tilde{p}u + \tilde{q}d\right\}$$

algebra

$$= \frac{S_n}{(1+r)^n}\left\{\frac{\tilde{p}u+\tilde{q}d}{1+r}\right\}$$

since $\tilde{p}u+\tilde{q}d = 1+r$.

$$= \frac{S_n}{(1+r)^n}. \blacksquare$$

Proof 2. There's an alternate proof in Shreve.

Example. Consider an N-period binomial model with $u = 1.25$, $d = 0.8$, $r = 0.1$. Stock S has price S_0. Let V be a derivative security with payoff given by $V_N(\omega) = \frac{1}{S_N(\omega)}$. Find the (arbitrage-free) price V_0.

Note that trying to do this with backward induction would suck. Let's try it a little:

At time N, there are N+1 possibilities: there are k heads & N-k tails, for $k=0,1,2,\dots,N$. The corresponding payouts are $\frac{1}{S_N(k\text{ H's }\&\text{ }N-k\text{ T's})} =$

$$= \frac{1}{u^k d^{N-k} S_0}, \text{ for each } k.$$

Going back one time to $t=N-1$, what's V_{N-1}(j heads, N-1-j tails)? For any $j=0,1,\dots,N-1$, the next step is either H or T. So we have

$$V_{N-1}(j\text{ H's}, N-1-j\text{ T's}) = \frac{1}{1+r}\left[\tilde{p}\frac{1}{u^{j+1}d^{N-1-j}S_0} + \tilde{q}\frac{1}{u^j d^{N-j}S_0}\right]$$

$$= \frac{1}{1+r}\frac{1}{S_0}\frac{1}{u^j d^{N-1-j}}\left[\frac{\tilde{p}}{u} + \frac{\tilde{q}}{d}\right]$$

$$= \frac{1}{1+r}\frac{1}{S_0}\frac{1}{u^{j+1}d^{N-j}}\left[\tilde{p}d + \tilde{q}u\right],$$

for all $j=0,1,2,\dots,N-1$.

By similar reasoning, if we have i heads at time N-2, the value should be

$$\frac{1}{1+r}\left[\tilde{p}V_{N-1}(i+1\text{ H's }\&\text{ }N-i) + \tilde{q}V_{N-1}(i\text{ H's }\&\text{ }N-1-i\text{ T's})\right]$$

$$= \frac{1}{1+r}\left[\tilde{p}\cdot\frac{\tilde{p}d+\tilde{q}u}{(1+r)u^{i+2}d^{N-i-1}S_0} + \tilde{q}\frac{\tilde{p}d+\tilde{q}u}{(1+r)u^{i+1}d^{N-i}S_0}\right]$$

$$= \frac{(\tilde{p}d+\tilde{q}u)^2}{(1+r)^2 u^{i+2}d^{N-i}S_0}.$$

OK, so we're kind of getting somewhere but this sucks. So let's do something smarter.

OK, so $(V_n)_{0\le n\le N} = \left(\frac{1}{S_n}\right)_{0\le n\le N}$ isn't a martingale under $\tilde{\mathbb{P}}$. But can we find some constants $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots, \alpha_N$ such that $\left(\frac{\alpha_n}{S_n}\right)_{0\le n\le N}$ is a martingale under $\tilde{\mathbb{P}}$.

Let $n \in \{0,1,\ldots,N-1\}$. Then

$$\tilde{\mathbb{E}}_n\left[\frac{\alpha_{n+1}}{S_{n+1}}\right] = \frac{\alpha_n}{S_n}$$

Since it's a martingale. So

$$\frac{\alpha_n}{S_n} = \tilde{\mathbb{E}}_n\left[\frac{\alpha_{n+1}}{S_{n+1}}\right] = \alpha_{n+1}\tilde{\mathbb{E}}_n\left[\frac{1}{S_{n+1}}\right]$$

$$= \alpha_{n+1}\left(\tilde{p}\frac{1}{uS_n} + \tilde{q}\frac{1}{dS_n}\right)$$

$$= \frac{\alpha_{n+1}}{S_n}\left(\frac{\tilde{p}}{u} + \frac{\tilde{q}}{d}\right)$$

$$= \frac{\alpha_{n+1}}{S_n}\left(\frac{\tilde{p}d + \tilde{q}u}{ud}\right)$$

So we need $\alpha_n = \frac{\alpha_{n+1}}{ud}(\tilde{p}d + \tilde{q}u)$, for all $n \in \{0,1,\ldots,N-1\}$.

May as well let $\alpha_0 = 1$. So then

$$\alpha_{n+1} = \left(\frac{ud}{\tilde{p}d+\tilde{q}u}\right)\alpha_n = \left(\frac{ud}{\tilde{p}d+\tilde{q}u}\right)^{n+1}.$$

So in our case,

$$\alpha_n = \left(\frac{(0.8)(1.25)}{\frac{2}{3}(0.8) + \frac{1}{3}(1.25)}\right)^n$$

$$= \left(\frac{3}{2.85}\right)^n$$

So we've shown $\left(\left(\frac{3}{2.85}\right)^n \frac{1}{S_n}\right)_{0\le n\le N}$ is a martingale (under $\tilde{\mathbb{P}}$).

Why is this useful? Well, $V_0 = \frac{1}{1.1^N}\tilde{\mathbb{E}}\left[\frac{1}{S_N}\right]$.

Since this is a martingale, we get

$$\frac{1}{S_0} = \frac{\alpha_0}{S_0} = \tilde{\mathbb{E}}_0\left[\left(\frac{3}{2.85}\right)^n \frac{1}{S_n}\right]$$

for all $n = 0, 1, \ldots, N$.

So $\frac{1}{S_0} = \left(\frac{3}{2.85}\right)^N \tilde{\mathbb{E}}_0\left[\frac{1}{S_N}\right] = \left(\frac{3}{2.85}\right)^N \tilde{\mathbb{E}}[1/S_N]$.

So,

$$V_0 = \frac{1}{1.1^N}\tilde{\mathbb{E}}[1/S_N] = \frac{1}{1.1^N}\left(\frac{2.85}{3}\right)^N \frac{1}{S_0}$$

$$= \left(\frac{2.85}{3.3}\right)^N \frac{1}{S_0}$$