MATH 321: Math Finance **Spring 2016**

Natasha Komarov Supplementary Material [1]

Hi and welcome to MATH 321: Math Finance! This document will supply some extra material that will be useful for us in addition to our textbook (*Stochastic Calculus for Finance I: The Binomial Asset Pricing Model* by Steven E. Shreve) [5]. I will update this document as needed throughout the semester with any necessary supplementary material.

In Section 1 we'll go over some of the basics of financial markets and the models we will be discussing this semester to analyze them. This will help us gather some context and motivation for the material before we head into Section 1.1 of the textbook (which introduces the one-period binomial model).

1 Introduction to Financial Modeling

Whenever we want to build a model to analyze a real life system, we have to choose where to fall on the complexity spectrum. Too far to the simple end of the spectrum and we have a model which allows us to draw powerful conclusions about something pretty inapplicable to reality; too far in the other direction and we have a very accurate model from which we can draw very few conclusions. The overarching goal of his class is to motivate, set up, and utilize a financial model which rests in the sweet spot of the complexity spectrum: simple enough to allow us to draw interesting conclusions, and complex enough to be useful in the real world. We'll begin with relatively simple models, but ones still complex enough for providing us with actual insights. By the end of the semester, we will take all of these insights and use them to help us derive the Black-Scholes-Merton model [1, 3], which is the actual model used in the real world for derivatives pricing.

1.1 Overview of Financial Markets, Replication, and Arbitrage

Financial markets revolve around the trading of assets. These assets include—among other things—shares of stock, bonds, commodities, and derivative securities. We will be focusing on securities trading. A *derivative security* (or simply *derivative* is an asset that **derives** its value from the performance of some more primitive asset. For

[1] Adapted from 21-270 course notes by William Hrusa and Dmitry Kramkov, Carnegie Mellon University, Pittsburgh, PA USA 15213

this definition to make any sense, we need to figure out exactly what we mean by the "value" of an asset, and just how this value ought to depend on the underlying asset. Generally, the most significant numbers associated with an asset are its payoff and its initial price. In order for this model to make any sense, the initial price should be determined in some systematic way by the payoff.

Pricing will be our focus in this class. There are two crucial concepts in the Black-Scholes-Merton model (as well as in every model we will consider on our way toward understanding the BSM model):

1. Replication is possible.

    This means that it should be possible **replicate** the payoff structure of the derivative security with a *replicating portfolio* of other assets.

2. Arbitrage is not possible.

    An *arbitrage strategy* is any trading strategy which requires no initial capital, has no risk of loss, and has a positive probability of gain.

It will be helpful to consider an example which illustrates both of these two concepts.

Example 1.1 *You arrive at an airport and notice two currency exchange booths; neither booth charges any fee for the exchange of currency. At Booth 1, you can exchange US dollars and British pounds (in either direction) at the rate of \$1.45 per £1. At Booth 2, you can exchange dollars and euros (in either direction) at the rate of \$1.10 per €1. But wait! Turns out there is a third booth, at which you can exchange British pounds and euros (in either direction). Assuming there is no fee, what should be the exchange rate at Booth 3?*

This is a fairly simple computation: based on the values given above,

$$\$1 = £0.68966 = €0.9091.$$

Therefore, dividing both sides of the second equality by 0.917 gives us that Booth 3 should offer the rate €1.3182 for £1. But what happens if it doesn't? Suppose that instead, Booth 3 offers the exchange rate €1.30 for £1. Is there some way to take advantage of this inconsistency? Suppose you start with \$100. You can exchange these \$100 for €90.91 at Booth 2, and immediately exchange that €91.74 for £69.93 at Booth 3. Now take this to Booth 1 and exchange this for \$101.40. You just made a 1.4% profit with very little effort and—more importantly—absolutely no risk. If you had no limitations on your starting amount, you could have made a great deal of money in this set of exchanges.

Does this strategy qualify as an arbitrage strategy? Not exactly as stated. We had to put in \$100 to start, and our definition of arbitrage necessitated no initial capital. However, assuming that it's possible to borrow money (for a short period of time) without incurring interest in our model, we could start by borrowing \$100, consequently using no initial capital of our own, and making \$1.40 without risking or paying anything.

Supposing that our model allows no arbitrage, Booth 3 has no choice but to set the rate we computed above of €1.3182 to £1. In this case, we can *replicate* the transaction at one booth using the other two booths. For example, if we wish to replicate the exchange of €131.82 for £100 at Booth 3, we could start with the same initial capital (€131.82), exchange this for dollars at Booth 2 (so we'd have \$145), and exchange the dollars for pounds at Booth 1 (ending up with £100). This is a replication strategy because it required the same initial capital and had the same payouts. □

The no-arbitrage assumption in a pricing model is sometimes referred to as *rational pricing*, which should help emphasize that it is a reasonable assumption. In real life, arbitrage opportunities do exist, but they are generally extremely short-lived and far more complex than the situation presented by the three currency exchange booths in Example 1.1. Consequently, if we are looking for some reasonable way to price a derivative security, the intuition behind rational pricing says that we can ignore these short-lived opportunities as they will be "arbitraged away" by arbitrage traders, and won't be reflected in the price of the security by the time we get to it. If you want to know more about arbitrage, this class won't really help you (as we will be assuming no arbitrage throughout), but there are many books, articles, and online resources that you may be interested in perusing (for example, [2, 4]).

1.2 Intro to Pricing Derivatives

Two of the most basic kinds of derivative securities which we will spend some time with in this class are put and call options. In general,

Definition 1.2 *A call option on a stock gives the option holder the right (but not the obligation) to buy one share of the stock on or before a specified date T (called the maturity, exercise, or expiration date) for a specified price K (called the strike price).*

Definition 1.3 *A put option on a stock gives the option holder the right (but not the obligation) to sell one share of the stock on or before a specified date T (called the maturity, exercise, or expiration date) for a specified price K (called the strike price).*

Definition 1.4 *A European option allows the holder to exercise the option only at* *the maturity date, but not before. In contrast, an American option can be exercised at any point after its purchase up to and including the maturity date.*

Let's take a European call in a very simple model and compute its rational, or arbitrage-free, price. Suppose that there is a single stock, with current stock price \$10 per share, and in one year the price will either go up to \$15 per share (with probability 1/3) or will go down to \$8 per share (with probability 2/3). Besides the single stock, the only other element in our model is a bank, which offers 2% interest, compounded annually. We can borrow money from the bank.

Example 1.5 *Find today's ($t = 0$) price for a European call on the stock described above in the given model with strike price $K = 12$ and maturity date one year from now ($t = 1$).*

A fundamental concept in this class is that **PRICING = REPLICATION**. (It's in bold and all caps, so you know it's **REALLY IMPORTANT**.) When we price derivatives, we will do it by creating a replicating portfolio. So let's try that now. There are only two situations: in situation (1) the stock price goes up to \$15, and consequently the call is worth \$3 to the holder (because she can exercise the option to purchase one share of the stock for \$12 and immediately sell it for \$15); in situation (2) the stock price drops to \$8 and the call expires worthless (because there is no reason for the holder to purchase the stock at \$12 when it only costs \$8 on the open market). So the payouts are either (1) \$3 or (2) \$0. We need to create a replicating portfolio which only uses the underlying stock and the bank and yields the same payouts in the same situations as the call option. When we can determine the initial capital required for this replicating portfolio, we will have figured out the initial price of the call option.

Let's call the number of shares of stock purchased (at time $t = 0$) in the replicating portfolio α and the amount of money invested in the bank at $t = 0$ β. Then in situation (1), our portfolio is worth $15\alpha + 1.02\beta$ at time 1. In situation (2), it is worth $8\alpha + 1.02\beta$. In order for this to replicate the call option, we must have

$$\begin{aligned} 15\alpha + 1.02\beta &= 3 \qquad (1) \\ 8\alpha + 1.02\beta &= 0 \qquad (2) \end{aligned}$$

Solving these equations yields $\alpha = 3/7, \beta = -400/119$. This means that we need to buy 3/7 share of stock and borrow \$400/119 from the bank. Since the current price of the stock is \$10, it will cost us $\$30/7 \approx 4.29$ to buy the stock. The initial capital for this strategy is therefore $10\alpha + \beta = 30/7 - 400/119 = 110/119 \approx 0.92$. Since there is no arbitrage in our model, it must be the case that the arbitrage-free price of the call is equal to the the amount of initial capital required for this replicating portfolio (since the call and the replicating portfolio are equivalent). Therefore, the price, V_0, of the call is \$0.92. $\square$

Note that this price never incorporated the risk (i.e. the probabilities we gave in the statement of the model conditions). It might have been tempting to try to say that the initial price should just be given by the expected value

$$(1/3)(3) + (2/3)(0) = 1 \tag{3}$$

but this would be incorrect. We will see more about how to think of computing the price as a computation of expected value when we talk about the *risk-neutral measure*. Intuitively, the problem with the computation in Equation 3 is that if we were relying on the actual probabilities of the stock value going up or down, we would have to take each individual's risk preferences into account (if you are less averse to risk than I am, you should be willing to pay more than I am willing to pay for a security which might give you a lot of money with low probability and costs you something with high probability), which would result in an inconsistent pricing model. This doesn't mean that these probabilities are irrelevant, however. I'll leave you with that cryptic comment for now, and we'll see more when we talk about the *risk-neutral measure*.

1.3 Assumptions for the Model

OK, so what is a *financial model*, exactly? Roughly speaking, a financial model is made up of:

- a set $\mathcal{S}$ of traded securities,
- a set $\mathcal{T}$ of possible trading times,
- some rules governing the actual trade of the securities,
- some mathematical assumptions about prices and how these prices change over time

The set $\mathcal{T}$ contains some initial time, t_0, so that $t \geq t_0$ for all $t \in \mathcal{T}$. Generally, we'll have $t_0 = 0$ and we'll assume there is at least one $t > t_0$ in $\mathcal{T}$ (or else this would be a super boring model). Note that the set $\mathcal{T}$ is generally clear from context, so we won't always have to state it explicitly.

The price of an asset at a given time is a function which depends on, among other things, the aforementioned time. What this means for us is that if today is time $t = 0$, I don't actually know the arbitrage-free price for which some asset should sell at time $t = 3$. This price depends on what happens in the market between now and then. Therefore the price of an asset at a future time is a *random variable* on a *probability space*. (We'll talk more in depth about probability in the future.)

We will assume that all models we discuss this semester do not allow arbitrage, and do allow replication. To this end, we will assume during this class that all financial models have a risk-free investment: a bank account. (It is risk-free because we assume the banks will not default on loans; thankfully, this has almost always been the case in our national history, so it's not too wild an assumption.) We will assume that there are no limits on how much money can be borrowed from or invested in the bank (this is a less reasonable assumption when taken to extremes, but the extremes are rarely going to be what we're really studying). The bank will be associated with some interest rate, r, which is the same for both investing and borrowing. We make this assumption so that we can think of borrowing money as investing a negative amount of money, which will make computations far more reasonable. (So depositing -30 dollars in the bank is the same as borrowing 30 dollars from the bank.) In addition to borrowing money, the model also allows for borrowing assets: that is, *short-selling*. (Returning the borrowed asset by purchasing it and giving it to the party from which it was borrowed is called *covering* or *closing out* the short position.)

Note that based on our assumptions about arbitrage and replication, the following observations can be made.

1. The initial capital for all replicating strategies for a given security must be equal (and this common value must be equal to the arbitrage-free price of the security).
2. If we add a replicable security to our arbitrage-free model, then the new model will be arbitrage free if and only if the inital price of this new security is equal tot he initial capital of a replicating strategy.

(Note that if either of these observations were not true, the very concept of an arbitrage-free price would not make sense.

We will be interested most often in investment strategies which are *self-financing*. This means that capital cannot be added or removed from the portfolio after the initial time (although reallocation of assets is acceptable; e.g. selling a stock and investing the proceeds from the sale in the bank). Note that based on our assumptions about arbitrage and replication, if any two self-financing strategies have the same terminal time and the same terminal capitals in all possible states at this terminal time, then the initial capitals of the two strategies must be the same. (If a security makes a payment at more than one time, then we cannot require that replicating strategies be self-financing. We'll cross that bridge when we come to it.)

We will ignore transaction costs (which includes bid-ask spread). This means that if a certain security has price V_0, this is the price for which it is to be bought as well as sold. We will also ignore any and all taxes.

1.4 Initiation Exercises

To help solidify the concepts in this section, try out the following exercises.

Example 1.6 *Consider a financial model with two times, $t = 0$ and $t = 1$. T here is a bank with one-period interest rate $r = 0.2$. There are two basic (risky) securities: a stock S that pays no dividents, and a European put P on the stock with expiration date $T = 1$ and strike price $K = \$10$. The initial stock price is $S_0 = \$10$ per share and the initial put price is $P_0 = \$2.50$ per option. There are three possible outcomes, $\omega_1, \omega_2, \omega_3$:*

- *In outcome ω_1, the stock price at $t = 1$ will be $S_1 = \$4$.*
- *In outcome ω_2, the stock price at $t = 1$ will be $S_1 = \$21$.*
- *In outcome ω_3, the stock price at $t = 1$ will be $S_1 = \$18$.*

That is, $S_1(\omega_1) = 4, S_1(\omega_2) = 21, S_1(\omega_3) = 18$.

1. *Let C denote a European call option on the stock with expiration date $T = 1$ and strike price $K = 10$. Find the aribtrage-free price C_0 of the call.*
2. *Let V be a derivative security that pays $\$1$ at time 1 if $S_1 > 10$ and pays nothing otherwise. Find the aribtrage-free price V_0 of the security.*

Try these two exercises out yourself before you read through the solution here!

Did you try it out?

DID YOU?

OK, here's the solution.

1. So, it's no accident that the strike price is the same here as it was for the corresponding put option. We'll see more about this concept when we deal with put-call parity, which will help us realize how to easily price a put with the same strike price and maturity date as a call with the same strike price and maturity date and whose initial price is already known to us (and vice versa). For now, let's use good old replication.

   Unlike in Example 1.5, we have a bank and not one but two risky securities. We'll want to purchase some number α shares of the stock, some number β put options, and invest some amount $\$\gamma$ in the bank. If we do that, the capital, X_1, in our replicating portfolio at time 1 will be given by

$$X_1 = \alpha S_1 + \beta P_1 + 1.2\gamma \tag{4}$$

Note that based on the values for S_1 in each of the three possible outcomes given in the stipulations of the problem, we know that $P_1(\omega_1) = 6$ and $P_1(\omega_2) = P_1(\omega_3) = 0$. We can also compute just as easily that $C_1(\omega_1) = 0, C_1(\omega_2) = 11, C_1(\omega_3) = 8$. So if we want to replicate this call using the bank, the stock, and the put (all that's available to us in this model), we need to have $X_1(\omega_i) = C_1(\omega_i)$ for all $i = 1, 2, 3$. Substituting the appropriate values into Equation (4) gives

$$\begin{aligned} 0 &= 4\alpha + 6\beta + 1.2\gamma && (5) \\ 11 &= 21\alpha + 0\beta + 1.2\gamma && (6) \\ 8 &= 18\alpha + 0\beta + 1.2\gamma && (7) \end{aligned}$$

Now we just have to solve Equations (5), (6), and (7) simultaneously. We get $\alpha = 1, \beta = 1, \gamma = -\frac{25}{3}$. This means we have to buy one share of stock, one put, and borrow $\$\frac{25}{3}$ from the bank at time 0 to replicate one call. Therefore, $C_0 = S_0 + P_0 - \frac{25}{3} = \$\frac{25}{6}$.

Note that the fact that $\alpha = \beta = 1$ was not just a coincidence. This is a result of the put-call parity mentioned above. Keep this in the back of your mind for when we discuss this notion further in a few days!

2. Before we get started, let's see if we can develop some (very basic) intuition about what we should see here. Should the price V_0 be lower or higher than P_0? C_0? Can we give any other range we know it ought to fall between? If we think about this a little bit, we should immediately be able to realize that since this security will only ever pay out either \$0 or \$1, the price should be less than \$0.83 (since you can get \$1 at $t = 1$ risk-free by investing about \$0.83 in the bank at time 0, and there is some risk that V won't even pay you a whole dollar). The price should also, of course, be positive (otherwise you'd be getting a possible something for nothing, which would be arbitrage). These observations don't give a great estimate on the price, but they do give us a sanity check.

   OK, let's actually compute it now. Once again, let's write out the payouts in each of the three cases:

$$V_1(\omega_1) = 0, V_1(\omega_2) = V_1(\omega_3) = 1$$

   So now the system of equations we should have looks like

$$\begin{aligned} 0 &= 4\alpha + 6\beta + 1.2\gamma && (8) \\ 1 &= 21\alpha + 0\beta + 1.2\gamma && (9) \\ 1 &= 18\alpha + 0\beta + 1.2\gamma && (10) \end{aligned}$$

Solving this system (by, for instance, subtracting (10) from (9) to get α, plugging this into (9) or (10) to get β, and plugging both of these values into (8) to get γ), we get $\alpha = 0, \beta = -\frac{1}{6}, \gamma = \frac{25}{3}$. Therefore

$$V_0 = 0 - \frac{1}{6}(10) + \frac{25}{3} = \frac{5}{12} \approx \$0.42.$$

(Note that, thankfully, this value passes our sanity check test.)

2 Optimal investment in the N-period binomial model

In this section, we'll give a little more detail on optimal investment beyond what is provided in Section 3.3 (Capital Asset Pricing Model) in the textbook.

As usual, consider an N-period binomial model with up factor u, down factor d, and interest rate r. We assume that the reference measure $\mathbb{P}$ is a binomial product measure with probability of heads equal to p and probability of tails equal to q.

Let $\Lambda > 0$ be given. We consider first an economic agent who wishes to invest the amount $X_0 = \Lambda$ at time 0. The agent will adjust the portfolio at each time $n = 1, 2, \cdots, N-1$, but no capital will be added or removed at these times. Since the portfolio will be self-financing, the terminal capital X_N must satisfy

$$\Lambda = \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[X_N].$$

The agent wishes to adjust the portfolio in such a way that $\mathbb{E}[U(X_N)]$ will be maximized. Here $\mathbb{E}$ stands for expected value with respect to the reference (or actual) probability measure $\mathbb{P}$ and $U : (0, \infty) \to \mathbb{R}$ is the agent's utility function.

Let $\mathcal{X}_\Lambda$ be the set of all random variables X_N on Ω satisfying $X_N(\omega) > 0$ for all $\omega \in \Omega$ and $\tilde{\mathbb{E}}[X_N] = \Lambda(1+r)^N$. Notice that $\mathcal{X}_\Lambda$ is precisely the set of all terminal capitals of self-financing strategies having initial capital Λ and strictly positive terminal capital.

Theorem 3.3.7: Assume that U is twice differentiable on $(0, \infty)$ and that $U'(x) > 0$, $U''(x) < 0$ for all $x > 0$. Let $\hat{X}_N \in \mathcal{X}_\Lambda$ be given. Then

$$\mathbb{E}[U(X_N)] \leq \mathbb{E}[U(\hat{X}_N)] \quad \text{for all } X_N \in \mathcal{X}_\Lambda \tag{11}$$

if and only if there exists $\mu \in \mathbb{R}$ such that

$$U'(\hat{X}_N(\omega)) = \mu Z(\omega) \quad \text{for all } \omega \in \Omega, \tag{12}$$

where Z is the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

Proof: Assume that (11) holds. Let $\mathcal{Y}$ denote the set of all random variables Y_N on Ω such that $\tilde{\mathbb{E}}[Y_N] = 0$. Observe that $\mathcal{Y}$ is a linear space (or vector space) over $\mathbb{R}$. (In fact, $\mathcal{Y}$ is simply the set of all terminal capitals of self-financing strategies having zero initial capital.) Let $Y_N \in \mathcal{Y}$ be given. Then we may choose an open interval I with $0 \in I$ such that

$$\hat{X}_N(\omega) + yY_N(\omega) > 0 \;\text{ for all } \omega \in \Omega,\; y \in I.$$

Since

$$\tilde{\mathbb{E}}[\hat{X}_N + yY_N] = \tilde{\mathbb{E}}[\hat{X}_N] + y\tilde{\mathbb{E}}[Y_N] = \Lambda + 0 = \Lambda,$$

we conclude that

$$\hat{X}_N + yY_N \in \mathcal{X}_\Lambda \text{ for all } \; y \in I. \tag{13}$$

Define $g : I \to \mathbb{R}$ by

$$g(y) = \mathbb{E}[U(\hat{X}_N + yY_N)] \;\text{ for all } y \in I,$$

and notice that

$$g(0) = \mathbb{E}[U(\hat{X}_N)]. \tag{14}$$

In view of (11), (13), and (14) we know that

$$g(y) \leq g(0) \;\text{ for all } y \in I.$$

Since g is differentiable on I we must have

$$g'(0) = 0.$$

A simple calculation shows that

$$g'(y) = \mathbb{E}[U'(\hat{X}_N + yY_N)Y_N] \;\text{ for all } y \in I. \tag{15}$$

Putting $y = 0$ in (15) and recalling that $Y_N \in \mathcal{Y}$ was arbitrary, we conclude that

$$\mathbb{E}[U'(\hat{X}_N)Y_N] = 0 \text{ for all } Y_N \in \mathcal{Y}. \tag{16}$$

Since the elements of $\mathcal{Y}$ are characterized in terms of risk-neutral expected value, it is natural to change measure in (16) to obtain

$$\tilde{\mathbb{E}}\left[\frac{U'(\hat{X}_N)}{Z}Y_N\right] = 0 \text{ for all } Y_N \in \mathcal{Y}. \tag{17}$$

Let us put

$$\mu = \tilde{\mathbb{E}}\left[\frac{U'(\hat{X}_N)}{Z}\right], \tag{18}$$

and observe that

$$\frac{U'(\hat{X}_N)}{Z} - \mu \in \mathcal{Y}. \tag{19}$$

It follows easily from (17) that

$$\tilde{\mathbb{E}}\left[\left(\frac{U'(\hat{X}_N)}{Z} - \mu\right)Y_N\right] = \tilde{\mathbb{E}}\left[\frac{U'(\hat{X}_N)}{Z}Y_N\right] - \mu\tilde{\mathbb{E}}[Y_N] = 0. \text{ for all } Y_N \in \mathcal{Y}. \tag{20}$$

Combining (19) and (20) we find that

$$\tilde{\mathbb{E}}\left[\left(\frac{U'(\hat{X}_N)}{Z} - \mu\right)^2\right] = 0. \tag{21}$$

It follows easily from (21) that

$$\frac{U'(\hat{X}_N)}{Z} - \mu = 0,$$

which yields (12).

To prove the converse implication, let $\mu \in \mathbb{R}$ be given and assume that (12) holds. Let $X_N \in \mathcal{X}_\Lambda$ be given. We need to show that

$$\mathbb{E}[U(X_N)] \leq \mathbb{E}[U(\hat{X}_N)]. \tag{22}$$

Since

$$\hat{X}_N(\omega) > 0 \quad \text{and} \quad X_N(\omega) > 0 \quad \text{for all } \omega \in \Omega$$

we may choose an open interval J such that $[0,1] \subset J$ and

$$yX_N(\omega) + (1-y)\hat{X}_N(\omega) > 0 \quad \text{for all } y \in J.$$

Define $f : J \to \mathbb{R}$ by

$$f(y) = \mathbb{E}[U(yX_N + (1-y)\hat{X}_N)] \quad \text{for all } y \in J, \tag{23}$$

and observe that

$$f'(y) = \mathbb{E}[U'(yX_N + (1-y)\hat{X}_N)(X_N - \hat{X}_N)] \quad \text{for all } y \in J, \tag{24}$$

$$f''(y) = \mathbb{E}[U''(yX_N + (1-y)\hat{X}_N)(X_N - \hat{X}_N)^2] \quad \text{for all } y \in J. \tag{25}$$

Since $X_N, \hat{X}_N \in \mathcal{X}_\Lambda$, we know that

$$\tilde{\mathbb{E}}(X_N - \hat{X}_N) = 0. \tag{26}$$

Using (24), change of measure, (12), and (26) we conclude that

$$f'(0) = \mathbb{E}[U'(\hat{X}_N)(X_N - \hat{X}_N)] = \tilde{\mathbb{E}}\left[\frac{U'(\hat{X}_N)}{Z}(X_N - \hat{X}_N)\right] = \tilde{\mathbb{E}}[\mu(X_N - \hat{X}_N)] = 0.$$

It follows easily from (25) and our assumption on U'' that $f''(y) \leq 0$ for all $y \in J$. We conclude that $f(y) \leq f(0)$ for all $y \in J$; in particular

$$f(1) \leq f(0).$$

Since $f(1) = \mathbb{E}[U(X_N)]$ and $f(0) = \mathbb{E}[U(\hat{X}_N)]$, we have established the desired conclusion. $\square$

Remark 3.3.8: In order to find the optimal terminal capital, we must solve the pair of equations

$$U'(\hat{X}_N) = \mu Z, \quad \tilde{\mathbb{E}}[\hat{X}_N] = \Lambda(1+r)^N \tag{27}$$

for a real number μ and a random variable $\hat{X}_N : \Omega \to \mathbb{R}$. Observe that we must have $\mu > 0$. Our assumptions on U imply that U' has an inverse function $F : \mathcal{R}(U') \to (0, \infty)$ where $\mathcal{R}(U')$ denotes the range of U'. We can rewrite (27) as

$$\hat{X}_N = F(\mu Z), \quad \tilde{\mathbb{E}}[F(\mu Z)] = \Lambda(1+r)^N. \tag{28}$$

Logarithmic Utility

Perhaps the most popular utility function of all (at least with mathematicians) is the logarithmic utility function U defined by

$$U(X) = ln\ x \text{ for all } x > 0, \tag{29}$$

for which we have

$$U'(x) = \frac{1}{x} \text{ forall } x > 0.$$

In order to solve the optimal investment problem for this utility function, we must find $\mu \in \mathbb{R}$ and $\hat{X}_N : \Omega \to \mathbb{R}$ satisfying

$$\frac{1}{\hat{X}_N} = \mu Z, \quad \tilde{E}[\hat{X}_N] = \Lambda(1+r)^N. \tag{30}$$

Using the fact that $\tilde{\mathbb{E}}[Z^{-1}] = 1$, it is straightforward to show that the unique solution to (30) is given by

$$\mu = \frac{1}{\Lambda(1+r)^N}, \quad \hat{X}_N = \frac{\Lambda(1+r)^N}{Z}.$$

Since our strategy is self-financing the capital $\hat{X}_n$ in the optimal portfolio at time n is given by

$$\hat{X}_n = \frac{1}{(1+r)^{N-n}} \tilde{\mathbb{E}}_n[\hat{X}_N] = \Lambda(1+r)^n \tilde{\mathbb{E}}_n[Z^{-1}] = \frac{\Lambda(1+r)^n}{Z_n}. \tag{31}$$

Since we have an explicit expression for Z_n in terms of $p, q, \tilde{p}, \tilde{q}$ and the number of heads and tails in the first n coin tosses, it is possible to write an explicit expression for $\hat{X}_n(\omega_1, \cdots, \omega_n)$:

$$\hat{X}_n(\omega_1, \cdots, \omega_n) = \Lambda(1+r)^n \left(\frac{p}{\tilde{p}}\right)^{\#H(\omega_1,\cdots,\omega_n)} \left(\frac{q}{\tilde{q}}\right)^{\#T(\omega_1,\cdots,\omega_n)}. \tag{32}$$

Notice that

$$\hat{X}_{n+1}(\omega_1, \cdots, \omega_n, H) = (1+r)\left(\frac{p}{\tilde{p}}\right)\hat{X}_n(\omega_1, \cdots, \omega_n) \tag{33}$$

and

$$\hat{X}_{n+1}(\omega_1, \cdots, \omega_n, T) = (1+r)\left(\frac{q}{\tilde{q}}\right)\hat{X}_n(\omega_1, \cdots, \omega_n). \tag{34}$$

These two equations imply that the number of shares of stock in the optimal portfolio at each time n will be proportional to the total capital divided by the stock price. This tells us that the percentage of capital that should be invested in stock at each time n is independent of n and independent of the results of the first n coin tosses.

Let $\alpha(\omega_1, \cdots, \omega_n)$ denote the percentage of capital in stock between time n and time $n+1$ in the optimal portfolio. Observe that

$$\alpha_n(\omega_1, \cdots, \omega_n) = \frac{\hat{\Delta}_n(\omega_1, \cdots, \omega_n) S_n(\omega_1, \cdots, \omega_n)}{\hat{X}_n(\omega_1, \cdots, \omega_n)}, \tag{35}$$

where

$$\hat{\Delta}_n(\omega_1, \cdots, \omega_n) = \frac{\hat{X}_{n+1}(\omega_1, \cdots, \omega_n, H) - \hat{X}_n(\omega_1, \cdots, \omega_n, T)}{S_{n+1}(\omega_1, \cdots, \omega_n, H) - S_{n+1}(\omega_1, \cdots, \omega_n, T)}. \tag{36}$$

Combing (33), (34), and (35) we arrive at

$$\alpha(\omega_1, \cdots, \omega_n) = \frac{(1+r)}{u-d}\left[\frac{p}{\tilde{p}} - \frac{q}{\tilde{q}}\right]. \tag{37}$$

Example 3.3.9: Consider an N-period binomial model with $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $p = \frac{2}{3}$, and $q = \frac{1}{3}$. Suppose that an investor with strictly positive initial capital Λ uses the utility function given by (18) and wishes to construct a self-financing portfolio for which the expected utility of the terminal capital is as large as possible.

We compute the quantity in (37):

$$\frac{(1+r)}{u-d}\left[\frac{p}{\tilde{p}} - \frac{q}{\tilde{q}}\right] = \frac{5}{4}\frac{1}{(2-\frac{1}{2})}\left[2\frac{2}{3} - 2\frac{1}{3}\right] = \frac{5}{9}.$$

This tells us that the investor should invest $\frac{5}{9}\Lambda$ in stock initially, and each time $n = 1, 2, \cdots, N-1$, the portfolio should be adjusted so that $\frac{5}{9}$ of the total capital is invested in stock. The reader should compare this Example with Problem 3 on Assignment 1 and with Example 3.3.2 in Shreve. $\square$

References

[1] F. Black and M. Scholes (1973). "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* **81** (3), 637—654.

[2] R. Dubil (2004), *An Arbitrage Guide to Financial Markets*, John Wiley & Sons, West Sussex, UK.

[3] R. C. Merton (1973). "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science (The RAND Corporation)* **4** (1), 141—183.

[4] R. Roll and S. A. Ross (1984). "The Arbitrage Pricing Theory Approach to Strategic Portfolio Planning." *Financial Analysts Journal* **20** (2), 122—131.

[5] S. E. Shreve (2004), *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*, Springer-Verlag, New York, USA.