Hunter Vs. Mole

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Pursuit games on graphs

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- Graphs on which a cop can win (i.e. capture) in finite time are called **cop-win**.
- Game takes no more than $n-4$ moves on cop-win graphs with $n \geq 7$ (see [1, 2]).

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- **Definition**. A graph is **dismantlable** if it has a sequence of **corners** (a.k.a. vertices dominated by another vertex) that leads to the trivial graph.

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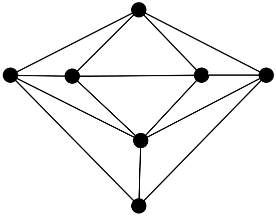

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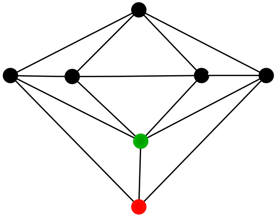

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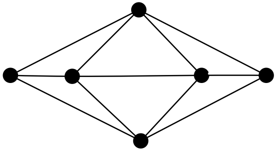

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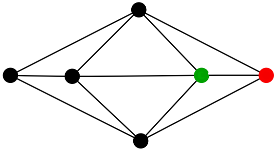

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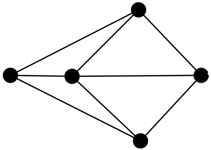

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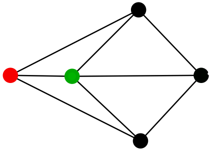

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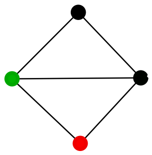

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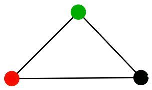

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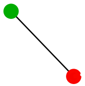

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Theorem

A graph is cop-win if and only if it is **dismantlable**.

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Bonus: Dismantlable graphs turn out to be important in unexpected areas: e.g. statistical physics.

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Theorem

A graph is hunter-win if and only if it is a lobster.

Characterization & Optimal Strategy

Definition

A **lobster** is a tree containing a path P such that all vertices are within distance 2 of P.

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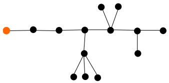
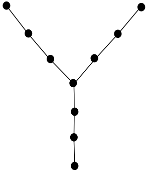


A lobster.



Not a lobster.

Characterization & Optimal Strategy

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Lobsters are hunter-win.

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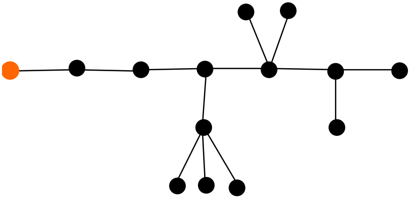

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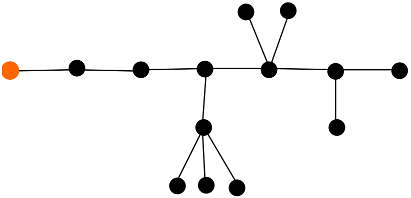

Define an **odd** (resp. **even**) **mole** to be a mole who starts at an odd (resp. even) distance from the marked vertex.

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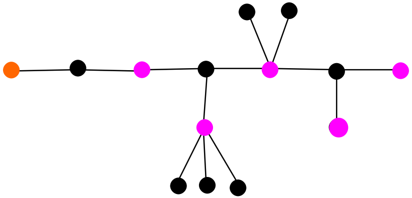

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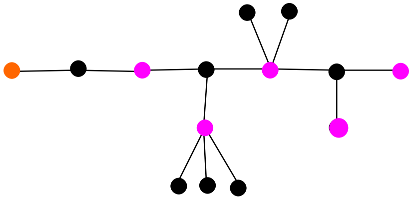
Proof by picture. After hunter's 1^{st} step:



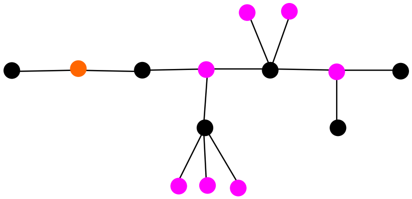
Orange vertex = hunter's position
Purple vertex = even mole's possible positions

Characterization & Optimal Strategy

Lemma

Lobsters are hunter-win.

Proof by picture. After hunter's 2^{nd} step:



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Characterization & Optimal Strategy

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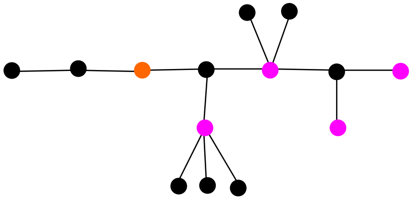
Lobsters are hunter-win.

Proof by picture. After hunter's 3^{rd} step:



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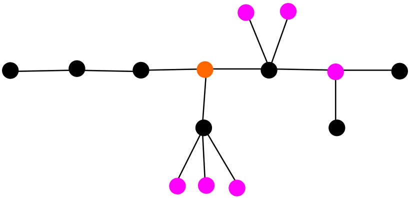
Characterization & Optimal Strategy

Lemma

Lobsters are hunter-win.

Proof by picture. After hunter's 4^{th} step:



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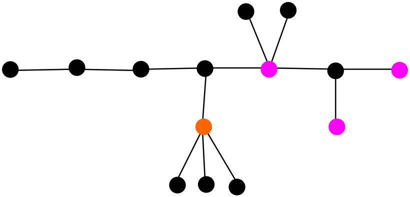
Characterization & Optimal Strategy

Lemma

Lobsters are hunter-win.

Proof by picture. After hunter's 5^{th} step:



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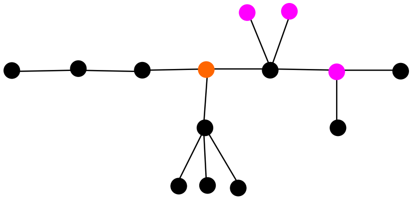
Characterization & Optimal Strategy

Lemma

Lobsters are hunter-win.

Proof by picture. After hunter's 6^{th} step:



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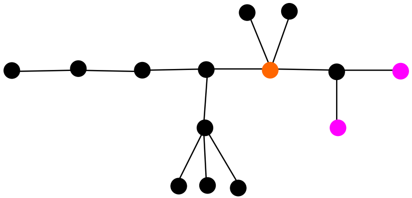
Characterization & Optimal Strategy

Lemma

Lobsters are hunter-win.

Proof by picture. After hunter's 7^{th} step:



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Characterization & Optimal Strategy

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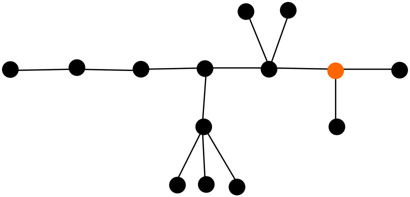
Lobsters are hunter-win.

Proof by picture. After hunter's 8^{th} step:



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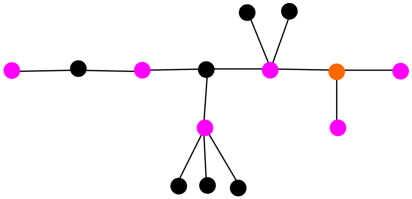

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Characterization & Optimal Strategy

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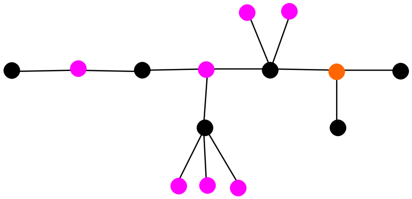
Proof by picture. After hunter's 9^{th} step:



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Characterization & Optimal Strategy

Lemma

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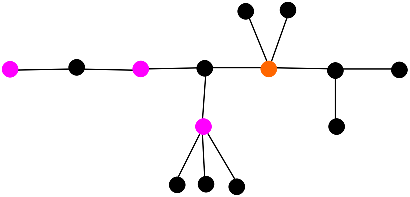
Proof by picture. After hunter's 10^{th} step:



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Characterization & Optimal Strategy

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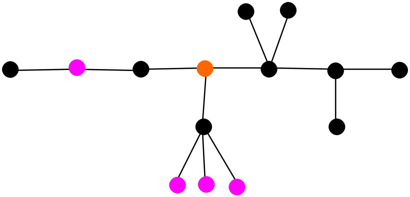
Proof by picture. After hunter's 11^{th} step:



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Characterization & Optimal Strategy

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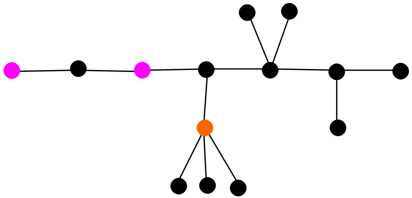
Proof by picture. After hunter's 12^{th} step:



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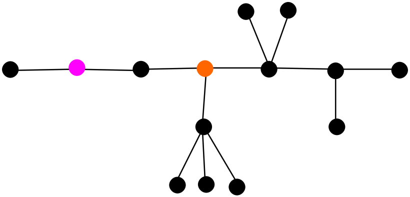

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Characterization & Optimal Strategy

Lemma

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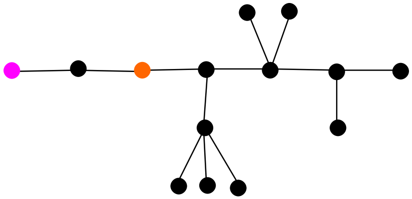
Proof by picture. After hunter's 14^{th} step:



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Characterization & Optimal Strategy

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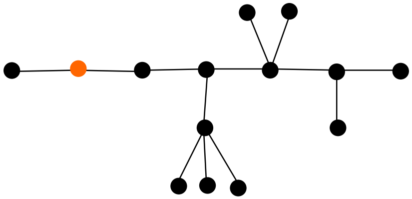
Proof by picture. After hunter's 15^{th} step:



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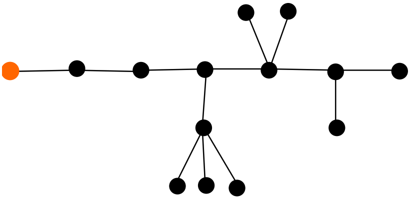

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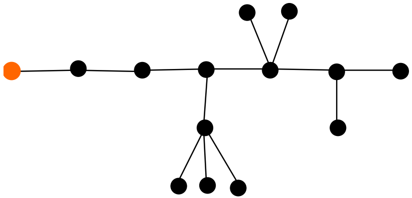
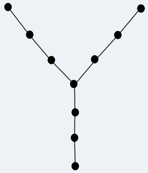


This is an optimal strategy for the hunter, by the way—even though she only went from vertex to **adjacent** vertex!

Characterization

Lemma

A graph G is a lobster if and only if it is a tree that doesn't contain the three-legged spider:



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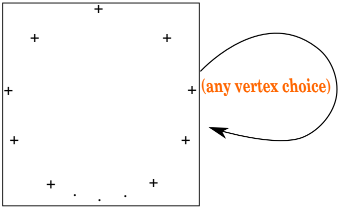

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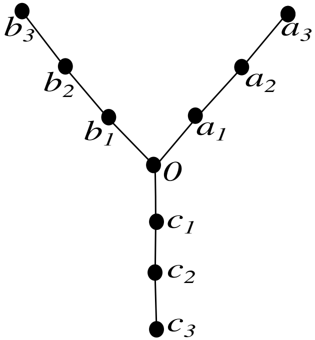

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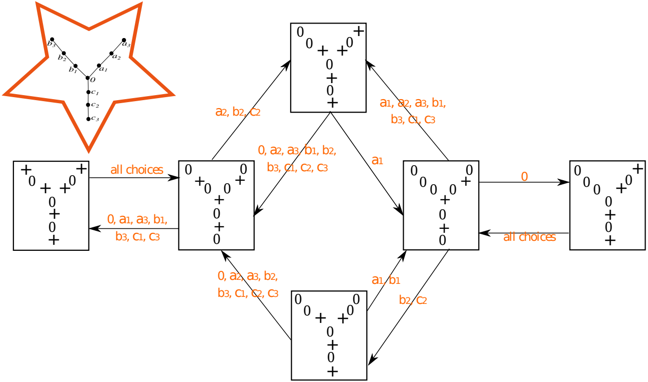

Conclusion

Theorem

A graph is hunter-win if and only if it is a lobster.

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Variation: A less restricted mole

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- Can be modeled by the original game, with loops at vertices where the mole is allowed to sit.

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- Can be modeled by the original game, with loops at vertices where the mole is allowed to sit.
- **Answer**: This is pretty bad news for the hunter!

Variation: A less restricted mole

- **Question**: What if the mole is allowed to stay put?
- Can be modeled by the original game, with loops at vertices where the mole is allowed to sit.
- **Answer**: This is pretty bad news for the hunter!

Lemma

Any graph with a loop at two or more vertices is mole-win.

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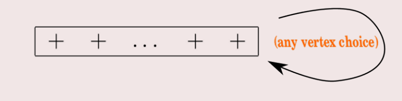

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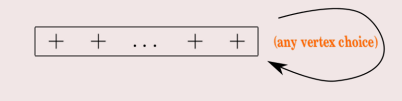

And any graph with a loop at two or more vertices contains such a path as a subgraph. □

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A graph containing exactly one loop is hunter-win if and only if it is a lobster and the loop is close to an endpoint of the central path.

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In particular, lobsters with a single loop are mole-win if and only if they contain one of the following graphs as a subgraph:

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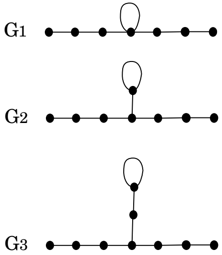

Variation: More hunters

- **Question**: Can we characterize all 2-hunter-win graphs?

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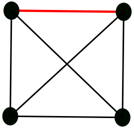
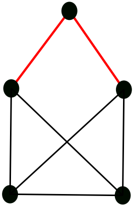


Mole-win



2-hunter-win

Variation: More hunters

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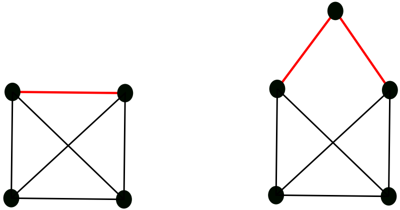

Mole-win *2-hunter-win*

- This is current undergraduate work with Zachary Greenberg (senior math major, CMU).

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Thank you!