7.5 Taking-and-Breaking Games

There are many natural variations on NIM obtained by modifying the legal moves. For example, sometimes a player, in addition to taking counters, might also be permitted to split the remaining heap into two (or sometimes more) heaps. These rule variants yield a rich collection of Taking-and-Breaking games that are discussed in *WW* [BCG01].

After NIM, Taking-and-Breaking games are among the earliest and most studied impartial games; however, by no means is everything known. To the contrary, much of the field remains wide open. Values in games such as GRUNDY'S GAME (choose a heap and split it into two different sized heaps) and Conway's COUPLES ARE FOREVER (choose a heap and split it but heaps of size 2 are not allowed to be split) have been computed up to heaps of size 11×10^9 and 5×10^7, respectively, yet there is no complete analysis for these games.

In Taking-and-Breaking variants, the legal moves may vary with the size of the heap and the history of the game. For example, the legal moves might be, "a player must take at least one-quarter of the heap and no more than one-half," or "a player must take between $n/2$ and $n+3$, where n is the number taken on the last move." For an example, see Problem 11. Games whose allowed moves depend on the history of the game are typically more difficult to analyze, but when the legal moves are independent of the history (and of moves in other heaps), then the game is a disjunctive sum and we only need analyze games that have a single heap!

Definition 7.28. For a given Taking-and-Breaking game G, let $\mathcal{G}(n)$ be the nim-value of the game played with a heap of size n. The *nim-sequence* for the game is $\mathcal{G}(0)$, $\mathcal{G}(1)$, $\mathcal{G}(2)$, ...

In order to automate the process for finding (and proving) the nim-sequences of selected Taking-and-Breaking games, we need to address two main questions:

1. What types of regularities occur in nim-sequences?
2. When do we know that some regularity observed in a nim-sequence will repeat for eternity?

There are three types of regularities that have been observed in many nim-sequences to which we can answer the second question. These are listed in the next definition but we only consider two of the three, those that are periodic and arithmetic periodic, in this book. The reader interested in split-periodicity should read [HN03, HN04].

Definition 7.29. A nim-sequence is

- *periodic* if there is some $l \geq 0$ and $p > 0$ so that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq l$;

- *arithmetic periodic* if there is some $l \geq 0$, $p > 0$, and $s > 0$ so that $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ for all $n \geq l$;[3] and
- *sapp regular* (or *split arithmetic periodic/periodic*) if there exist integers $l \geq 0$, $s > 0$, $p > 0$, and a set $S \subseteq \{0, 1, 2, \ldots, p-1\}$ such that for all $n \geq l$,
$$\mathcal{G}(n+p) = \begin{cases} \mathcal{G}(n) & \text{if } (n \bmod p) \in S, \\ \mathcal{G}(n) + s & \text{if } (n \bmod p) \notin S. \end{cases}$$

The subsequence $\mathcal{G}(0), \mathcal{G}(1), \ldots, \mathcal{G}(l-1)$ is called the *pre-period* and its elements are the *exceptional values.* When l and p are chosen to be as small as possible, subject to meeting the conditions of the definition, we say that l is the *pre-period length* and p is the *period length*, while s is the *saltus.* If there is no pre-period the nim-sequence is called *purely periodic, purely arithmetic periodic, purely sapp regular,* respectively.

Exercise 7.30. Match each sequence on the left one-to-one to a category on the right:

1231451671 . . .	periodic
1123123123 . . .	purely periodic
1122334455 . . .	sapp regular
0123252729 . . .	arithmetic periodic
0120120120 . . .	purely sapp regular
1112233445 . . .	purely arithmetic periodic

In each case, identify the period and (when non-zero) the saltus and pre-period.

7.6 Subtraction Games

Definition 7.31. A *subtraction game* denoted SUBTRACTION(S), is played with heaps of counters and a set S of positive integers. A move is to choose a heap and remove any number of counters provided that number is in S.

- If $S = \{a_1, a_2, \ldots, a_k\}$ is finite, we have a *finite subtraction game*, which we denote SUBTRACTION($a_1, a_2, \ldots, a_k$).
- If, on the other hand, $S = \{1, 2, 3, \ldots\} \setminus \{a_1, a_2, \ldots, a_k\}$ consists of all the positive integers except a finite set, we have an *all-but subtraction game*, denoted ALLBUT($a_1, a_2, \ldots, a_k$).

In Example 7.25, SUBTRACTION(1, 2, 4) was shown to be periodic. On the other hand, ALLBUT() is another name for NIM and is arithmetic periodic with saltus 1.

[3] Reminder: + means normal, not nimber, addition!

Finite subtraction games

The next table gives the first 15 values of the nim-sequence for several subtraction games SUBTRACTION(S).

S	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
{1,2,3}	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2
{2,3,4}	0	0	1	1	2	2	0	0	1	1	2	2	0	0	1
{3,4,5}	0	0	0	1	1	1	2	2	0	0	0	1	1	1	2
{3,4,6,10}	0	0	0	1	1	1	2	2	2	0	3	3	1	4	0

It is not surprising that the nim-sequence for SUBTRACTION$(1, 2, 3)$ is purely periodic with values $\dot{0}12\dot{3}$. (*Note*: we use dots to indicate the first and last values in the period.) The nim-sequence for SUBTRACTION$(2, 3, 4)$ is $\dot{0}0112\dot{2}$ and $\dot{0}0011122\dot{2}$ for SUBTRACTION$(3, 4, 5)$. The pattern is not yet evident for SUBTRACTION$(3, 4, 6, 10)$, but if we pushed on we would eventually find that the nim-sequence is $00011122203314\dot{0}020131\dot{2}$.

Calculating nim-sequences with a Grundy scale

The calculation of nim-sequences for subtraction games can be done by a computer (CGSuite, Maple or Mathematica, for example, or any spreadsheet program) but there is an easy, hand technique as well. Use two pieces of graph paper to construct what is known as a *Grundy scale*. Here, we will work with the example SUBTRACTION$(3, 4, 6, 10)$.

To begin, take two sheets of lined or graph paper. On each sheet make a scale, with the numbers marked in opposite directions. One of the scales is marked with △s to indicate the positions of numbers in the subtraction set; you can put a ▲ to indicate 0. The other sheet will be used to record the nim-values:

0	1	2	3	4	5	6	7

	△				△		△	△			▲
11	10	9	8	7	6	5	4	3	2	1	0

As you fill in nim-values on the top scale, slide the bottom scale to the right. Shown below is the calculation $\mathcal{G}(9)$; take the mex of the positions marked by arrows. In this case, $\text{mex}(1, 1, 2) = 0$, so we will fill in a 0 for $\mathcal{G}(9)$, now indicated by the ▲:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1	1	1	2	2	2	**0**					

	△				△		△	△			▲
11	10	9	8	7	6	5	4	3	2	1	0

Similarly, $\mathcal{G}(13) = \text{mex}(1, 2, 0, 3) = 4$:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	0	1	1	1	2	2	2	0	3	3	1	**4**						

	△				△		△	△			▲
11	10	9	8	7	6	5	4	3	2	1	0

Exercise 7.32. Make a photocopy of the last Grundy scale, make a cut to separate the bottom and top portions, and use it to calculate more of the nim-sequence of SUBTRACTION$(3, 4, 6, 10)$. For more practice, try using Grundy scales for the examples in problem 2 until you are comfortable.

Periodicity of finite subtraction games

After working out a few finite subtraction games, it will come as no surprise that their nim-sequences are always periodic.

Theorem 7.33. *The nim-sequences of finite subtraction games are periodic.*

Proof: Consider the finite subtraction game SUBTRACTION$(a_1, a_2, \ldots, a_k)$ and its nim-sequence. From any position there are at most k legal moves. So, using Observation 7.22, $\mathcal{G}(n) \leq k$ for all n.

Define $a = \max\{a_i\}$. Since $\mathcal{G}(n) \leq k$ for all n there are only finitely many possible blocks of a consecutive values that can arise in the nim-sequence. So we can find positive integers q and r with $a \leq q < r$ such that the a values in the nim-sequence immediately preceding q are the same as those immediately preceding r. Then $\mathcal{G}(q) = \mathcal{G}(r)$ since

$$\mathcal{G}(q) = \text{mex}\{\mathcal{G}(q - a_i) \mid 1 \leq i \leq k\} = \text{mex}\{\mathcal{G}(r - a_i) \mid 1 \leq i \leq k\} = \mathcal{G}(r).$$

In fact, for such q and r and all $t \geq 0$, $\mathcal{G}(q+t) = \mathcal{G}(r+t)$. This is easily shown by induction — we have just seen the base case, and the inductive step is really just an instance of the base case translated t steps forwards. Now set $l = q$ and $p = r - q$ and we see that the above says that for all $n \geq l$, $\mathcal{G}(n + p) = \mathcal{G}(n)$; that is, that the nim-sequence is periodic. □

This proof shows that the pre-period and period lengths are at most $(k+1)^a$. However, this is generally a wild overestimate, and using the following corollary the values of the period and pre-period lengths can usually be determined by computer:

Corollary 7.34. *Let* $G = \text{SUBTRACTION}(a_1, a_2, \ldots, a_k)$ *and let* $a = \max\{a_i\}$. *If* l *and* p *are positive integers such that* $\mathcal{G}(n) = \mathcal{G}(n+p)$ *for* $l \leq n < l + a$, *then the nim-sequence for* G *is periodic with period length* p *and pre-period length* l.

Proof: See Problem 5. □

That is, given conjectured values of l and p, it suffices to inspect the values of $\mathcal{G}(n)$ for $n \in \{l, l+1, \ldots, l+p+a-1\}$ to confirm the periodicity! It is then rote for a computer to identify the smallest pre-period and period given by the corollary.

Applying the corollary to the games in Table 7.6 we see that for:

SUBTRACTION$(1, 2, 3)$ we have $l = 0$, $p = 4$ and $a = 3$, and these values can be confirmed by inspection of $\mathcal{G}(n)$ for $n \in \{0, \ldots, 6\}$;

SUBTRACTION$(2, 3, 4)$ $l = 0$, $p = 6$ and $a = 4$, inspect $\mathcal{G}(n)$ for $n \in \{0, \ldots, 9\}$;

SUBTRACTION$(3, 4, 5)$ $l = 0$, $p = 8$ and $a = 5$, inspect $\mathcal{G}(n)$, $n \in \{0, \ldots, 12\}$; and

SUBTRACTION$(3, 4, 6, 10)$ $l = 14$, $p = 7$ and $a = 10$, inspect $\mathcal{G}(n)$, $n \in \{14, \ldots, 30\}$.

Exercise 7.35. Use a Grundy scale to compute values of SUBTRACTION$(2, 4, 7)$ until you have enough to apply Corollary 7.34. For more practice, see Problem 2.

Two of the main questions, which still attract researchers, are:

1. As a function of the a_i, how long can the period of SUBTRACTION$(a_1, a_2, \ldots, a_k)$ be?
2. Find general forms for the nim-sequence for SUBTRACTION(a_1, a_2, a_3).

Many games with different subtraction sets are actually the same in the sense that they have the same nim-sequence. For example, in the game SUBTRACTION(1), the first player wins precisely when there is an odd number of counters in the heap. The odd-sized heaps only have moves to even-sized heaps; even-sized heaps only have moves to odd-sized heaps and the end position consist of heaps of size 0; that is, even. Therefore, a heap with n-counters is in $\mathcal{N}$ if n is odd, otherwise it is in $\mathcal{P}$. Adjoining any odd numbers to the subtraction set does not change this argument and it is easy to show that the nim-sequence of any game SUBTRACTION$(1, \text{odds})$ is $\dot{0}\dot{1}0101$. A more general version of this analysis is sufficient to prove:

Theorem 7.36. *Let* $G = \textsc{subtraction}(a_1, a_2, \ldots, a_k)$ *be purely periodic with period* p. *Let* $H = \textsc{subtraction}(a_1, a_2, \ldots, a_k, a_1 + mp)$ *for* $m \geq 0$, *then* G *and* H *have the same nim-sequence.*

Proof: See Problem 6. □

All-but-finite subtraction games

The next table gives the first 15 values of the nim-sequence for ALLBUT(S) for several sets S:

S	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
{1,2,3}	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3
{2,3,4}	0	1	0	1	0	1	2	3	2	3	2	3	4	5	4
{1,2,8,9,10}	0	0	0	1	1	1	2	2	2	3	0	3	4	1	4

Grundy scales can still be used for the all-but subtraction games. This time, the small arrows mark the heaps that are not options.

Exercise 7.37. Use a Grundy scale to find the first 20 terms in the nim-sequence of ALLBUT$(1, 3, 4)$.

These values certainly do not look periodic, for $\mathcal{G}(n)$ appears to steadily increase with n, as confirmed by the following lemma:

Lemma 7.38. *Let* $G = \textsc{allbut}(a_1, a_2, \ldots, a_k)$ *and* $a = \max\{a_i\}$. *Then* $\mathcal{G}(n+t) > \mathcal{G}(n)$ *for all* $t > a$.

Proof: Since any option from n is also an option from $n+t$, $\mathcal{G}(n+t) \geq \mathcal{G}(n)$. Additionally, n is an option from $n+t$, so $\mathcal{G}(n)$ occurs as the nim-value of an option from $n+t$ and thus we do not have equality. □

However, as we will see, all-but subtraction games are arithmetic periodic. From the table above, we might conjecture the following (we denote the saltus information in parentheses with a + sign):

S	nim-sequence
{1,2,3}	$\dot{0}00\dot{0}(+1)$
{2,3,4}	$\dot{0}1010\dot{1}(+2)$
{1,2,8,9,10}	$0001112\dot{2}3\dot{0}(+1)$

Proving a nim-sequence is arithmetic periodic is typically more difficult than proving periodicity. Nonetheless, we can parallel the work done for finite subtraction games, proving first that ALLBUT subtraction games are arithmetic periodic, and then identifying how one (a person or a computer) can automatically confirm the arithmetic periodicity.

Theorem 7.39. *Let $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$. Then the nim-sequence for G is arithmetic periodic.*

Proof: Lemmas 7.41 and 7.43, motivated and then proved below, yield the theorem. □

The proof of this theorem is more technical than that of Theorem 7.33, but in broad strokes is similar. In the proof of Theorem 7.33, we first argued that nim-values cannot get too large, and that therefore some sufficiently long sequence of nimbers must repeat. Once a sufficiently long sequence appears identically twice, an induction argument establishes that those two sequences remain in lock-step ad infinitum.

With arithmetic periodicity, the repetition we seek is in the *shape* of a sequence rather than its values as shown in Figure 7.1 on page 152. As you read through the proofs, keep in mind that two subsequences of nim-values, call them $(\mathcal{G}(n_0), \ldots, \mathcal{G}(n_0 + c))$ and $(G(n_0'), \ldots, \mathcal{G}(n_0') + c)$ have the same *shape* if

1. the two subsequences differ by a constant:
$$\mathcal{G}(n_0') - \mathcal{G}(n_0) = \mathcal{G}(n_0' + 1) - \mathcal{G}(n_0 + 1) = \cdots = \mathcal{G}(n_0' + c) - \mathcal{G}(n_0 + c),$$
2. or equivalently, both subsequences move up and down the same way: for all $0 \le i < c$,
$$\mathcal{G}(n_0 + i + 1) - \mathcal{G}(n_0 + i) = \mathcal{G}(n_0' + i + 1) - \mathcal{G}(n_0' + i).$$

It will turn out that the base case for our inductive proof will require a repetition of length about $2a$, where $a = \max\{a_i\}$. We show that such a repetition exists in the next two lemmas.

Lemma 7.40. *Let $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$, and define $a = \max\{a_i\}$. For all $n \ge a$,*
$$k - a \le \mathcal{G}(n + 1) - \mathcal{G}(n) \le a - k + 1.$$

Proof: Fix $n > a$ and let $X \subseteq \{\mathcal{G}(0), \mathcal{G}(1), \mathcal{G}(2), \ldots, \mathcal{G}(n - 1)\}$ be the nim-values of the options of n.[4] Now, since $\mathcal{G}(n)$ is the mex of X, we know that $\{0, 1, 2, \ldots, \mathcal{G}(n) - 1\} \subseteq X$. Further, play in $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$ prohibits moves to k of the top a heap sizes. Hence, one of $\{\mathcal{G}(0), \ldots, \mathcal{G}(n-a-1)\}$, say $\mathcal{G}(m)$, must be at least $\mathcal{G}(n) - 1 - (a - k)$, for only $a - k$ of the terms from X can appear among $\{\mathcal{G}(n - a), \ldots, \mathcal{G}(n) - 1\}$. Further, m and all the options from m are also options from $n + 1$. So we have

$$\begin{aligned} \mathcal{G}(n + 1) &> \mathcal{G}(m), \text{ and so} \\ \mathcal{G}(n + 1) &\ge \mathcal{G}(n) - (a - k). \end{aligned}$$

[4] In other words, $X = \{\mathcal{G}(n - \alpha) \mid \alpha \notin \{a_1, \ldots, a_k\}\}$.

Similarly, for the second inequality, one of $\{\mathcal{G}(0), \dots, \mathcal{G}(n-a-1)\}$ is at least $\mathcal{G}(n+1) - 2 - (a-k)$, and it and its options are options of n. So,

$$\mathcal{G}(n) \quad \geq \quad \mathcal{G}(n+1) - (a-k) - 1. \qquad \square$$

Lemma 7.41. *Let* $G = \text{ALLBUT}(a_1, a_2, \dots, a_k)$ *and* $a = \max\{a_i\}$. *There exist* n_0, n_0', s, *and* $p = n_0' - n_0 > 0$ *such that* $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ *for* $n_0 \leq n \leq n_0 + 2a$.

Proof: By Lemma 7.40, for all n, $\mathcal{G}(n+1) - \mathcal{G}(n)$ must lie between $k - a$ and $a - k + 1$. But there are only $2(a-k) + 2$ values in that range. Hence, setting $c = 2(a-k) + 2$, there are at most c^{2a} possible sequences of the form

$$\begin{pmatrix} \mathcal{G}(n+1) - \mathcal{G}(n), \\ \mathcal{G}(n+2) - \mathcal{G}(n+1), \\ \dots, \\ \mathcal{G}(n+2a) - \mathcal{G}(n+2a-1) \end{pmatrix} \tag{7.1}$$

and so eventually, for two values $n = n_0$ and $n = n_0'$, the two corresponding sequences are identical. The lemma follows. □

Exercise 7.42. Complete the algebra to confirm, "The lemma follows." In particular, given the two identical sequences satisfying (7.1), one with n_0', and one with n_0, you need to define p and explain why $\mathcal{G}(n+p) - \mathcal{G}(n)$ is a constant (call it $s = \mathcal{G}(n_0') - \mathcal{G}(n_0)$) for $n_0 \leq n \leq n_0 + 2a$. (The matching shapes in Figure 7.1 provide some intuition.)

The next lemma completes the inductive step of the proof of the theorem showing that once two sufficiently long sequences have the same shape, they are fated to continue in lock step.

Lemma 7.43. *Let* $G = \text{ALLBUT}(a_1, a_2, \dots, a_k)$ *and* $a = \max\{a_i\}$ *and suppose* $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ *for* $n_0 \leq n \leq n_0 + 2a$. *Then* $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ *for all* $n \geq n_0$.

In particular, $\mathcal{G}(n)$ has pre-period length $l = n_0$, period p, and saltus s, and one can identify and confirm the period and saltus by only inspecting the first $l + 2a + p + 1$ values; i.e., $\mathcal{G}(n)$ for $0 \leq n \leq l + 2a + p$.

Proof: Using induction, it suffices to prove $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ for $n = n_0 + 2a + 1$. Define the following quantities as shown in Figure 7.1: $n_0' = n_0 + p$, $n_1 = n_0 + a$, $n_2 = n_0 + 2a$, $n_1' = n_0' + a$, and $n_2' = n_0' + 2a$.

As we compute $\mathcal{G}(n') = \mathcal{G}(n+p)$ as the mex of the nim-values of its options, by Lemma 7.38, $\mathcal{G}(n')$ exceeds all $\mathcal{G}(m)$ for $m < n' - a - 1 = n_1'$. And we

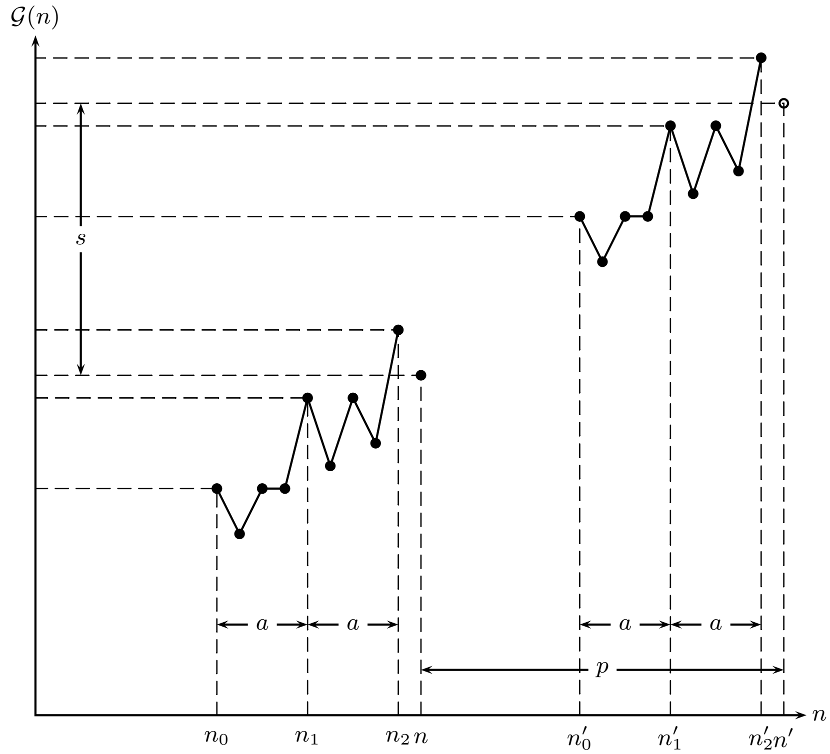

Figure 7.1. A diagram sketching the proof of arithmetic periodicity in ALLBUT subtraction games. For the game $\text{ALLBUT}(a_1, \ldots, a_k)$, $a = \max\{a_i\}$, s is the proposed saltus, p is the proposed period. We first find two sufficiently long sequences of nim-values with the same shape (but translated upward), and inductively prove that those sequences remain in lock-step.

know $\mathcal{G}(n') > \mathcal{G}(n'_1)$ in any event, so we can safely ignore $\mathcal{G}(m)$ for $m < n'_0$. In other words, $\mathcal{G}(n')$ is the minimum excluded value from $\{\mathcal{G}(n'_0), \ldots, \mathcal{G}(n'_2)\}$ *that exceeds* $\mathcal{G}(n'_1)$. Since the assignment of $\mathcal{G}(n')$ is unaffected by linear translation of those nim-values, $\mathcal{G}(n') - \mathcal{G}(n) = s$. □

This last lemma gives an automated method for testing when an all-but subtraction game nim-sequence has become arithmetic periodic. Although Figure 7.1 shows the two subsequences non-overlapping (suggesting that $p > 2a$), the proof is unaffected by overlap.

Exercise 7.44. Apply Lemma 7.43 to find the period length p and the saltus s of the game from Exercise 7.37. In particular, how many values of $\mathcal{G}(n)$ need computing to confirm the period and saltus? (*Hint*: The game is purely arithmetic periodic with period between 10 and 15.)

Exercise 7.45. We asserted at the start of this section that the nim-sequence for ALLBUT$(1, 2, 8, 9, 10)$ is given by $00011122\dot{2}3\dot{0}(+1)$. To apply Lemma 7.43, which values of $\mathcal{G}(n)$ need be confirmed to be confident of the nim-sequence?

As was seen in the table on page 149, some ALLBUT subtraction games have pre-periods. If you do Problems 12, 13, and 14. then you will have shown that the nim-values of ALLBUT games where s has cardinality 1 and 2 are purely arithmetic-periodic.

Frequently, the ALLBUT subtraction set can be reduced. While most such reductions remain specific to individual games, we do have one general reduction theorem.

Theorem 7.46. *Let $a_1 < a_2 < \cdots < a_k$ be positive integers, and let $b > 2a_k$. Then, the nim-sequences of* ALLBUT$(a_1, a_2, \ldots, a_k)$ *and* ALLBUT$(a_1, a_2, \ldots, a_k, b)$ *are equal.*

Proof: Let $\mathcal{G}(n)$ denote the nim-sequence of ALLBUT$(a_1, a_2, \ldots, a_k)$ and $\mathcal{G}'(n)$ that of ALLBUT$(a_1, a_2, \ldots, a_k, b)$. Certainly, $\mathcal{G}(n) = \mathcal{G}'(n)$ for $n < b$ since the options in the two games are identical to that point. Suppose inductively that the two nim-sequences agree through $n - 1$, and consider $\mathcal{G}(n)$ and $\mathcal{G}'(n)$. Since the options of ALLBUT$(a_1, a_2, \ldots, a_k, b)$ are a subset of those of ALLBUT$(a_1, a_2, \ldots, a_k)$, the only possible way to have $\mathcal{G}(n) \neq \mathcal{G}'(n)$ would be if $\mathcal{G}'(n) = \mathcal{G}(n - b)$, since the latter is the only possible value that does not occur among the options of ALLBUT$(a_1, a_2, \ldots, a_k, b)$ but does occur among the options of ALLBUT$(a_1, a_2, \ldots, a_k)$. In order for this to be true it would also be the case that no option of n in ALLBUT$(a_1, a_2, \ldots, a_k, b)$ had value $\mathcal{G}(n - b)$.

However, consider $m = n - b + a_k$. By the inductive hypothesis $\mathcal{G}(m) = \mathcal{G}'(m)$. Moreover, all values smaller than $n - b$ are options from a heap of size m in ALLBUT$(a_1, a_2, \ldots, a_k)$, so $\mathcal{G}(m) \geq \mathcal{G}(n - b)$. As m is an option of n in ALLBUT$(a_1, a_2, \ldots, a_k, b)$, in order to avoid a contradiction we would require that $\mathcal{G}(m) > \mathcal{G}(n - b)$. But then, m would have an option m' in ALLBUT$(a_1, a_2, \ldots, a_k)$ with $\mathcal{G}(m') = \mathcal{G}(n - b)$. Since $m' \neq n - b$ and $m' < n - a_k$, m' is also an option of n in ALLBUT$(a_1, a_2, \ldots, a_k, b)$. This contradiction establishes the desired result. □

For more on all-but-finite subtraction games see [Sieg06].

Kayles and kin

KAYLES is played with a row of pins standing in a row. The players throw balls at the pins. The balls are only wide enough to knock down one or two adjacent pins.

† † †   † † † † † †   †

Above is a game in progress. Would you like to take over for the next player?

KAYLES can also be thought of as a game played with heaps of counters and a player is allowed to take one or two counters and possibly split the remaining heap into two. KAYLES is clearly a Taking-and-Breaking game but maybe it is too hard right now. We will work up to it with a couple of simpler variants.

First, suppose we play a game where a player is only allowed to split-the-heap into two non-empty heaps; no taking. We saw in Example 1.10 of Chapter 1, that this is SHE LOVES ME SHE LOVES ME NOT disguised. That is,

$$\mathcal{G}(n) = \begin{cases} 0 & \text{if } n = 0, \\ 0 & \text{if } n > 0 \text{ is odd}, \\ 1 & \text{if } n > 0 \text{ is even}. \end{cases}$$

In particular, moves from an even heap are to even + even $= * + * = 0$ or odd + odd $= 0$, while moves from an odd heap are to even + odd $= * + 0 = *$.

The version where you are allowed to split-*or*-take-1-and-must-split (into non-empty heaps) needs a bit more care. For a heap of n, the splitting move gives the options $n - i$ and i for $i = 1, 2, \ldots, n-1$ and the take-1-and-split move leaves the positions $n - 1 - j$ and j, $j = 1, 2, \ldots, n-2$. To find the value of the game, we need to find the mex of

$$\begin{aligned} &\{\mathcal{G}(i) \oplus \mathcal{G}(n-i) \mid i = 1, 2, \ldots, n-1\} \\ \cup \quad &\{\mathcal{G}(j) \oplus \mathcal{G}(n-1-j) \mid j = 1, 2, \ldots, n-2\}. \end{aligned}$$

The first few values are $\mathcal{G}(0) = 0$, $\mathcal{G}(1) = 0$, $\mathcal{G}(2) = \text{mex}\{\mathcal{G}(1) + \mathcal{G}(1)\} = 1$, $\mathcal{G}(3) = \text{mex}\{\mathcal{G}(2) \oplus \mathcal{G}(1)\} \cup \{\mathcal{G}(1) \oplus \mathcal{G}(1)\} = 2$. This becomes tedious very quickly and a machine will be very useful. If one is not available then a Grundy scale can help. This time both pieces of paper have the nim-values, one from left-to-right as before, the other has them from right-to-left. However, we have to use them twice and record the intermediate values. The next diagram shows the calculation of $\mathcal{G}(9)$. First, line up the scales for the split move, $8+1$, $7+2$, …, $1+8$. Clearly we only need go to $4+5$ since we repeat the calculations in reverse. Nim-sum the pairs of numbers and record:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	1	4	3	2						

			2	3	4	1	3	2	1	0	0
11	10	9	8	7	6	5	4	3	2	1	0

This gives $\{2, 2, 6, 2, 2, 6, 2, 2\}$ as the set of values for these options. Now line them up for the take-one-and-split move.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	1	4	3	2						

			2	3	4	1	3	2	1	0	0
11	10	9	8	7	6	5	4	3	2	1	0

This gives the set $\{3, 5, 3, 0, 3, 5, 3\}$ for these options. The least non-negative number that does not appear in either set is 1, so record $\mathcal{G}(9) = 1$ on both papers:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	1	4	3	2	**1**					

		1	2	3	4	1	3	2	1	0	0
11	10	9	8	7	6	5	4	3	2	1	0

Exercise 7.47. Continue using the Grundy scale to compute five more nim-values of the last game where you are allowed to split-*or*-split-and-take-one-but-always-leave-two-heaps.

For a fixed (finite) set S of positive integers, we can define the variant SPLITTLES(S) where a player is allowed to take away s, for some $s \in S$, from a heap of size at least s and possibly split the remaining heap.

In particular, KAYLES is just SPLITTLES$(1, 2)$.

What sort of regularities could the nim-sequences of SPLITTLES(S) have? In the few examples so far, the nim-values do not grow very quickly at all suggesting that the nim-sequences are periodic *but Nobody Knows.* All we know is that the nim-sequences are *not* arithmetic periodic [Aus76, BCG01]. It is believed that SPLITTLES(S) is periodic when S is finite, and for periodicity we do have an automatic check.

Theorem 7.48. *Fix* SPLITTLES(S) *with* $m = \max S$. *If there exists integers,* $l \geq 0$ *and* $p > 0$ *such that* $\mathcal{G}(n+p) = \mathcal{G}(n)$ *for* $l \leq n \leq 2l + 2p + s$*, then* $\mathcal{G}(n+p) = \mathcal{G}(n)$ *for all* $n \geq l$*. That is, the period persists forever.*

Proof: See Problem 15. □

Problems

1. Find the nim-sequences for SUBTRACTION(S), where $|S| = 2$ and $S \subseteq \{1, 2, 3, 4\}$.

2. Use a Grundy scale and Corollary 7.34 to compute the nim-sequences of

   (a) SUBTRACTION$(2, 3, 5)$;
   (b) SUBTRACTION$(3, 5, 8)$;
   (c) SUBTRACTION$(1, 3, 4, 7, 8)$.

3. Find the period for

   (a) ALLBUT$(1, 2, 3)$;
   (b) ALLBUT$(5, 6, 7)$;
   (c) ALLBUT$(3, 4, 6, 10)$.

4. A game is played like KAYLES, only you cannot bowl the end of a row of pins. In NIM language, you can take one or two counters from a heap and you must split that heap into two *non-empty* heaps. Using a Grundy scale, calculate the first 15 nim-values for this game.

5. Prove Corollary 7.34 on page 148.

6. Prove Theorem 7.36 on page 149.

7. (This is a generalization of NIM.) POLYNIM is played on polynomials with non-negative coefficients. A move is to choose a single polynomial and reduce one coefficient and arbitrarily change or leave alone the coefficients on the smaller powers in this polynomial — $3x^2+15x+3$ can be reduced to $0x^2 + 19156x + 2345678 = 19156x + 2345678$. Analyze POLYNIM. In particular, identify a strategy for determining when a position is a $\mathcal{P}$-position analogous to Theorem 7.12.

8. Find the nim-sequences for SUBTRACTION$(1, 2q)$ for $q = 1, 2, 3$. Find the form of the nim-sequence of SUBTRACTION$(1, 2q)$ for arbitrary q.

9. Show that the nim-sequence for SUBTRACTION$(q, q+1, q+2)$ is $\dot{0}0^{q-1}1^q2\dot{2}$ if $q > 1$. (As usual, x^b is x repeated b times.)

10. Analyze SUBTRACTION$(1, 2, 4, 8, 16, \ldots, 2^i, \ldots)$.

11. Analyze this variant of NIM: On any move a player must remove at least half the number of counters from the heap.[5]

12. Find the periods for ALLBUT(1), ALLBUT(2), and ALLBUT(3). Conjecture and prove your conjecture for the period of ALLBUT(q).

[5] The nim-sequence for the game in which no more than half can be removed has a remarkable self-similarity property: If you remove the first occurrence of each number in the nim-sequence, then the resulting sequence is the same as the original! See [Lev06].

13. Find the periods for ALLBUT$(1, 2)$, ALLBUT$(2, 3)$, and ALLBUT$(3, 4)$. Conjecture and prove your conjecture for the period of ALLBUT$(q, q + 1)$.

14. Find the nim-sequence for ALLBUT(q, r), $q < r$. (*Hint*: There are two cases $r = 2q$ and $r \neq 2q$.)

15. Prove Theorem 7.48 on page 155.

16. The rules of the game TURN-A-BLOCK are at the textbook website, www.lessonsinplay.com, and you can play the game against the computer. Determine a winning strategy for TURN-A-BLOCK. You should be able to consistently beat the computer on the *hard* setting at 3×3 and 5×3 turn-a-block (and even bigger boards!). You should be able to determine who should win from any position up to 5×5.