Cycles in Tournaments

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MAA Seaway Section
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joint work with John Mackey (Carnegie Mellon University)

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What we care about when we care about extremal theory

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- Problems of the form: maximize/minimize X, subject to Y.

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- Problems of the form: maximize/minimize X, subject to Y.
- Or: how does the maximum/minimum value of X (subject to Y) compare to the average?

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- We completely answer the question of the $k = 5$ case.

Cycles in tournaments

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Let E_k^n be the expected number of directed k-cycles in a random tournament on n vertices.

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Theorem (*Kendall & Smith 1940*)

The number of directed 3-cycles in a tournament is maximized at approximately

$$\frac{n^3}{24} \sim E_3^n.$$

Cycles in tournaments

Does this still happen when $n > 4$?

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Theorem (*Beineke & Harary 1965*)

The number of directed 4-cycles in a tournament is maximized at approximately

$$\frac{n^4}{48} = \frac{4}{3}E_4^n.$$

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(E.g. David Berman's PhD thesis & subsequent work: "The Number of 5-Cycles in a Tournament"—UPenn 1973)

Cycles in tournaments

Is $k = 3$ just a fluke? What happens for $k > 4$?
Attempts made for $k = 5$ for a while.
(E.g. David Berman's PhD thesis & subsequent work: "The Number of 5-Cycles in a Tournament"—UPenn 1973)
No graphs with asymptotically more 5-cycles than average found, but proof remains elusive.

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5-cycles in tournaments

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Theorem (*Komarov & Mackey 2014*)

The maximum number of directed 5-cycles in a tournament is asymptotically equal to $\frac{3}{4}\binom{n}{5} = E_5^n$.

5-cycles in tournaments: proof

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Definition. The **edge degree sequence** of a tournament $T = (V, E)$ is a sequence $(X_e)_{e \in E}$ of ordered 4-tuples $X_e = (A(e), B(e), C(e), D(e))$

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Definition. $\gamma(T, k)$ is the number of k-cycles in a given tournament T.

The 12 non-isomorphic tournaments on 5 vertices

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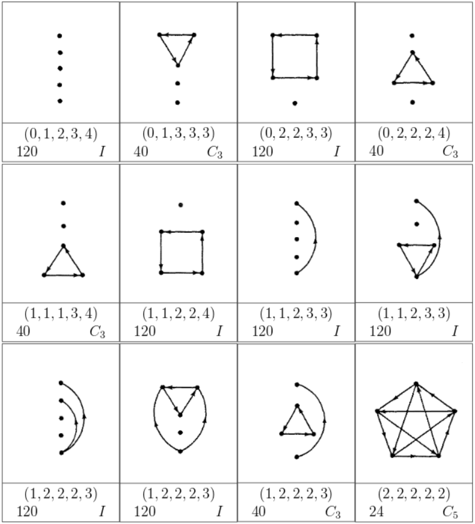

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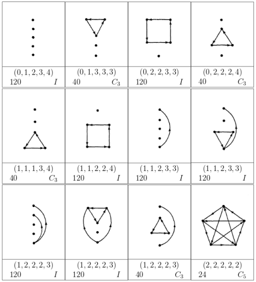

Call these T_1 through T_{12}.

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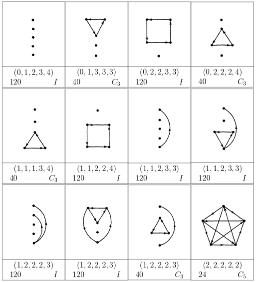

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$\gamma(T_1, 5) = \gamma(T_2, 5) = \cdots = \gamma(T_6, 5) = 0.$

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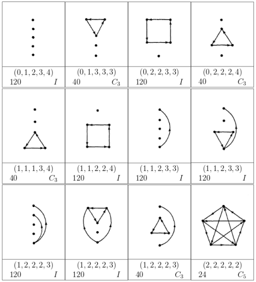

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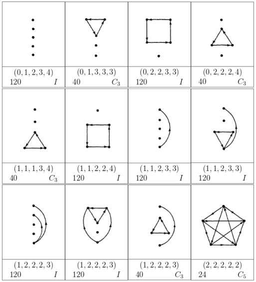

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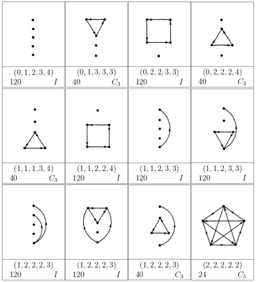

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$\gamma(T_{10}, 5) = \gamma(T_{12}, 5) = 2.$

$\gamma(T_{11}, 5) = 3.$

5-cycles in tournaments: proof

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Then
$\gamma(T,5) = T_7(T) + T_8(T) + T_9(T) + 2T_{10}(T) + 3T_{11}(T) + 2T_{12}(T)$.

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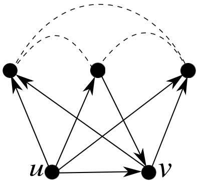
This counts the number of subtournaments of size 5 in T that look like:



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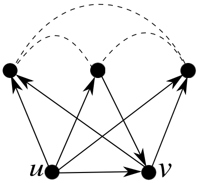

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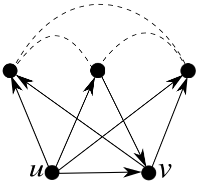

This occurs once in T_1, once in T_4, and twice in T_6.
So $\sum \binom{A(u,v)}{2} C(u,v) = T_1(T) + T_4(T) + 2T_6(T)$.

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Computing a few similar quantities (13, in all) in this way, then working some linear algebra magic, gives rise to a formula!

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Theorem (Komarov & Mackey 2014)

The number of 5-cycles in an n-tournament $T = (V, E)$ with edge degree sequence $(A(e), B(e), C(e), D(e))_{e \in E}$ is given by

$$\frac{3}{4}\binom{n}{5} - \frac{1}{8}\sum_{(u,v)\in E}\left[(C+D)(A-B)^2 + (A+B)(C-D)^2\right] + \frac{1}{4}\sum_{(u,v)\in E}(A+B)(C+D).$$

where $A = A(u,v)$, $B = B(u,v)$, $C = C(u,v)$, and $D = D(u,v)$.

5-cycles in tournaments: consequences

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Corollary

For all n-tournaments T,

$$\gamma(T,5) \leq \frac{3}{4}\binom{n}{5} + \frac{1}{4}\binom{n}{2}\left(\frac{n-2}{2}\right)^2$$

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$$= E_5^n + O(n^4)$$

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5-cycles in tournaments: consequences

Corollary

For all n-tournaments T,

$$\gamma(T,5) \geq \frac{3}{4}\binom{n}{5} - \frac{1}{4}(n-2)(n-3)\sum_{w\in V}\left(od(w) - \frac{n-1}{2}\right)^2 - \frac{n(n-2)(n^2-2n+2)}{8}$$

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So $\gamma(T,5) \sim E_5^n$ if and only if the standard deviation of the out-degrees is $o(n)$.

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Theorem (Savchenko 2015)

Among **regular** *tournaments R, the maximum number of k-cycles is asymptotically greater than E_k^n if $k \equiv 0 \mod 4$.*

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Among **regular** *tournaments R, the maximum number of k-cycles is asymptotically greater than* E_k^n *if* $k \equiv 0 \mod 4$.

Little else is known so far!

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Natural next steps:

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- Formula for 6-cycles in a tournament (given degree sequence of vertex 3-tuples).

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- For what k does the maximum number of k cycles approximately equal E_k^n?

Bigger cycles in tournaments

Natural next steps:

- Formula for 6-cycles in a tournament (given degree sequence of vertex 3-tuples).
- For what k does the maximum number of k cycles approximately equal E_k^n?
- Do regular tournaments have the most k-cycles?

Thank you!

References

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Historical context: undirected graphs

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Let G be a graph.

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Let G be a graph. Let $K_j(G)$, $I_j(G)$ be the number of complete subgraphs on j vertices in G and the number of independent sets of j vertices in G, respectively.

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Analogous question for undirected graphs: How does the minimum over all n-vertex graphs G of $K_j(G) + I_j(G)$ compare to the expected value for a random graph?

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Theorem (*Goodman 1959*)

The minimum value of $K_3(G) + I_3(G)$ over all n-vertex undirected graphs G is asymptotically equal to the expected number in a $p = \frac{1}{2}$ random graph, $\frac{1}{4}\binom{n}{3}$.

Erdös-Burr-Rosta Conjecture

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Conjecture (*Erdös 1962, Burr & Rosta 1980*)

For any graph G, and any integer $j \geq 3$,

$$I_j(G) + K_j(G) \text{ is minimized at about } \binom{n}{j} 2^{1-\binom{j}{2}}$$

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(which is the expected number of these in a $p = \frac{1}{2}$ random graph).

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The Erdös-Burr-Rosta Conjecture is false!

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Theorem (*Thomason 1989*)

The Erdös-Burr-Rosta Conjecture is false!
In fact, for each $j \geq 4$, there exists a family of graphs G such that

$$I_j(G) + K_j(G) > \binom{n}{j} 2^{1-\binom{j}{2}}$$