Containment: a Cops & Robber Variation

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- Some results on capture time [2, 5]
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- Cops and robber play with full information, moving alternately.
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- Each cop moves from edge to adjacent edge.
- Robber moves vertex to adjacent vertex; cannot use an occupied edge.
- For cops to win, they must contain the robber by occupying all edges incident to his position.
- What can we say about the containment number, $\xi(G)$, of a graph $G$?
Initial thoughts on $\xi(G)$
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For all $G$, $\xi(G) \geq \Delta(G)$.
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Examples:
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Examples:

- $C_n$
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**Examples:**

- $C_n$
- Graphs containing a universal vertex
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**Examples:**

- $C_n$
- Graphs containing a universal vertex
- Trees
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$C_k \square K_2$ is containable for all integers $k \geq 3$. 
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Game states:

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Game states:

- $P_t$: it’s the robber’s turn, two cops occupy parallel edges, the third cop is on one of the cycles; a shortest path from third cop to the cop on the same cycle has distance $t$ and contains the robber’s position
Another family of containable graphs

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- **\( Q_t \)**: it’s the robber’s turn, two cops occupy parallel edges, third cop is on an edge between the cycles such that
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- \( Q_t \): it’s the robber’s turn, two cops occupy parallel edges, third cop is on an edge between the cycles such that a shortest path from third cop to the other two cops has distance \( t \) and contains the robber’s position
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- Cops start at antipodal points; after robber’s placement, cops can move to be at state \(P_t\) with \(t < \frac{k}{2} - 1\).
- If game is in state \(P_t\) \((t > 0)\) then cops can move game into state \(Q_t\);
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- Cops start at antipodal points; after robber’s placement, cops can move to be at state \( P_t \) with \( t < \frac{k}{2} - 1 \).
- If game is in state \( P_t \) \( (t > 0) \) then cops can move game into state \( Q_t \); if game is in state \( Q_t \) then cops can move game into \( P_{t'} \) with \( t' < t \).
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**Proposition**

\[ C_k \square K_2 \text{ is containable for all integers } k \geq 3. \]

Cops can bring game to state \( P_0 \):
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Cops can bring game to state \( P_0 \):

![State P0 Diagram]

**Figure:** State \( P_0 \)
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Robber only has one option.
The cops then move to their endgame configuration:

Cops win on their next turn regardless of robber’s move.
Conjectural interlude
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Conjectural interlude

**Proposition**

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$T \square K_2$ is containable for all trees $T$. 
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Is $G \square K_2$ containable when $G$ is containable?
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Is \( G \square K_2 \) containable when \( G \) is containable?  
No.
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Is $G \Box K_2$ containable when $G$ is containable?

No. Counterexample: hypercubes.
Hypercubes

Proposition

\( Q_3 \) is containable.
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Proof.

\( Q_3 = C_4 \square K_2. \)
Hypercubes

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**Proof.**

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Proposition

\( Q_n \) is not containable for \( n \geq 4 \).
## Hypercubes

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In fact, at least $2n-2$ cops are required.
Hypercubes are not containable: Proof.

Robber is at $v$.  

$N(v) = \{v_1, \ldots, v_n\}$. 

We'll show that fewer than $2^n - 2$ cops cannot contain a lazy robber (who doesn't move if he doesn't have to).

Four cases:

1. 0 cops incident with robber.
2. Exactly 1 cop incident with robber.
3. Exactly $k$ cops incident with robber ($1 < k < n - 1$).
4. Exactly $n - 1$ cops incident with robber.
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After cops move, each cop can touch at most 2 of the vertices in \( \{v_1, \ldots, v_n\} \).
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After cops move, each cop can touch at most 2 of the vertices in \( \{v_1, \ldots, v_n\} \). Each one requires \( n \) cops that can move incident to it in order to prevent robber’s escape.
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After cops move, each cop can touch at most 2 of the vertices in \{v_1, ..., v_n\}. Each one requires \( n \) cops that can move incident to it in order to prevent robber’s escape. So at least \( n^2/2 \) cops are necessary in order for the cops to win on their move after the robber’s turn.
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Case 2: exactly 1 cop incident
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WLOG cop is on edge \( \{v, v_n\} \).
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WLOG cop is on edge \( \{v, v_n\} \). Every other cop can be adjacent to at most 2 of the vertices in \( \{v_1, \ldots, v_{n-1}\} \) on the next cop move.
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Case 3: Exactly $1 < k < n−1$ cops incident. WLOG, they’re at $\{v, v_{n−k+1}\}, \{v, v_{n−k+2}\}, \ldots, \{v, v_n\}$. To prevent escape to $v_1$, we need $n−1$ additional cops.
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Case 3: Exactly $1 < k < n-1$ cops incident. WLOG, they’re at $\{v, v_{n-k+1}\}, \{v, v_{n-k+2}\}, \ldots, \{v, v_n\}$. To prevent escape to $v_1$, we need $n-1$ additional cops. To also prevent escape to $v_2$, we need an additional $n-3$ cops (two of the cops preventing escape to $v_1$ can simultaneously be used for this purpose).
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This is already no less than $2n-2$. 
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Case 4: Exactly $n-1$ cops incident
An additional $n-1$ cops must be incident with robber’s one escape vertex.

□
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Hypercubes, continued

Proposition $\xi(Q^n) \leq (n^2)$ for all integers $n \geq 3$.

We can prove something stronger if we think about retracts.

Containment: a Cops & Robber Variation

Natasha Komarov
Proposition

\[ \xi(Q_n) \leq \binom{n}{2} \text{ for all integers } n \geq 3. \]
Proposition

\[ \xi(Q_n) \leq \left( \frac{n}{2} \right) \text{ for all integers } n \geq 3. \]

We can prove something stronger if we think about retracts.
Retracts

An induced subgraph $H \subset G$ is called a **retract** if there is a graph homomorphism $\phi : G \to H$ that restricts to the identity on $H$. 
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**Theorem**

*If $H \subseteq G$ is a retract of $G$, then $\xi(H) \leq \xi(G)$.***
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The analogous result holds for Cops & Robber, too.
Cubical retracts

Let $H \subset G$ be a retract under $\phi : G \to H$. 

Examples.
• Can retract $K_3$ onto $K_2$, but not cubically
• Can retract $C_4$ onto $K_2$ either as a cubical retract or not (either send both vertices outside the subgraph onto different vertices of $K_2$ or the same vertex of $K_2$)
• $Q_n + 1$ retracts cubically onto $Q_n \times \{0\} \sim Q_n$ by setting the last coordinate to 0.
Cubical retracts

Let $H \subset G$ be a retract under $\phi : G \rightarrow H$. $H$ is a **cubical retract** of $G$ if whenever $v \in V(G) \setminus V(H)$ is a vertex adjacent to $h \in H$, then we have $h = \phi(v)$.
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- $Q_{n+1}$ retracts cubically onto $Q^n \times \{0\} \cong Q^n$ by setting the last coordinate to 0.
# Theorem

Let $H \subseteq G$ be a cubical retract of $G$ under $\phi$. Then

$$\xi(G) \leq \max\{\xi(H), \xi(G - H)\} + dd(G, H) + \Delta(H) - 1$$

where $dd(G, H) = \max_{x \in H}(d_G(v) - d_H(v))$ is the degree discrepancy of $H$.
Cubical retracts

**Theorem**

Let $H \subseteq G$ be a cubical retract of $G$ under $\phi$. Then

$$\xi(G) \leq \max\{\xi(H), \xi(G - H)\} + dd(G, H) + \Delta(H) - 1$$

where $dd(G, H) = \max_{x \in H}(d_G(v) - d_H(v))$ is the **degree discrepancy** of $H$.

**Lemma**

Suppose that we are playing a containment game on a graph $G$ and that there are at least $c(G) + k - 1$ non-tail cops, then $k$ new tail cops can be attached to $R$. 

Containment: a Cops & Robber Variation

Natasha Komarov
Cubical retracts

Proof.

Let

\[ m = dd(G, H) + \Delta(H) + c(H) - 2 \]

and

\[ n = \max \{\xi(H), \xi(G - H)\} - c(H) + 1. \]

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Cubical retracts

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\[ n + c(H) - 1 = \max\{\xi(H), \xi(G - H)\} \] non-tail cops left.
Cubical retracts

Proof, cont’d.
Phase 2: these cops move until either the robber leaves $H$ or they contain him on $H$. 

Cubical retracts

Proof, cont’d.
Phase 2: these cops move until either the robber leaves $H$ or they contain him on $H$. If he leaves $H$, then the free $\max\{\xi(H), \xi(G - H)\}$ cops eventually contain him on $G - H$. 

Note: if $R$ ever moves from $G - H$ to $H$, he must move onto $\phi(R)$ (using the cubical property of the retract); we can fan out the $\text{dd}(G, H) + \Delta(H)$ tails on $\phi(R)$ to prevent $R$ from moving to any vertex other than the vertex of $G - H$ he came from.

The cops from Phase 2 can pursue $R$ as if he remained on the vertex he stood on before his move onto $H$.

Since there are at least $\xi(G - H)$ cops, they eventually contain the robber. □
Cubical retracts

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Cubical retracts

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Cubical retracts

Corollary

\[ \xi(Q_n) \leq \frac{n(n-1)}{2} \quad \text{for all } n \geq 3. \]
Cubical retracts

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**Proof.**

\( dd(Q_{n+1}, Q_n) = 1 \) and \( \Delta(Q_n) = n \), so

\[ \xi(Q_{n+1}) \leq \xi(Q_n) + 1 + n - 1 = \xi(Q_n) + n. \]

Use \( \xi(Q_3) = 3 \) and induction to get the desired result. \( \square \)
More containment number results
Proposition

If $G$ is a $\Delta$-regular ($\Delta > 2$) graph with girth at least 5, then $G$ is not containable.
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Example: Petersen graph (containment number $= 4$)
Background on Cops & Robbers

Preliminaries

Containability

Containment number

More containment number results

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Example: Petersen graph (containment number = 4)

**Proposition**

*If $G$ has girth at least 7 and is $\Delta$-regular ($\Delta > 2$), then $G$ is not containable by $\Delta + 1$ cops.*
Theorem

For all $G$, $c(G) \leq \xi(G) \leq \Delta(G) \gamma(G)$.

Proof sketch.

Lower bound: $\xi(G)$ cops play a Cops & Robber shadow game, with each cop staying on an endpoint of her Containment counterpart's edge; when the Containment game ends successfully for the cops, the Cops & Robber shadow game does too.

Upper bound: place a cop on each of the edges incident with each of the vertices in a dominating set of $G$. They can capture the robber in one step. □
More general result on containment number

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*For all graphs* \( G \), \( \xi(G) \leq \Delta(G)c(G) \).

This conjecture does hold “on average” in many random graphs [8].
**Containment number conjecture**

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This conjecture does hold “on average” in many random graphs [8].

$c(Q_n) = \lceil \frac{n+1}{2} \rceil$ (see [6]), so hypercubes provide an infinite class of examples where $\xi(G)$ is strictly less than $\Delta(G)c(G)$. 
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- What happens if the game is played on non-reflexive graphs? $\xi(T) = 1$ for all trees and the Petersen graph becomes containable. Non-reflexive containability should probably be defined as $\xi(G) = \delta(G)$. 
Thank you!
References


