Containment: a Cops & Robber Variation

Natasha Komarov

Department of Math, CS, and Stats
St. Lawrence University

University of Bordeaux
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- Some results on **capture time** [2, 5]

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- What can we say about the **containment number**, $\xi(G)$, of a graph G?

Initial thoughts on $\xi(G)$

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- Trees

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- If game is in state P_t $(t > 0)$ then cops can move game into state Q_t; if game is in state Q_t then cops can move game into $P_{t'}$ with $t' < t$.

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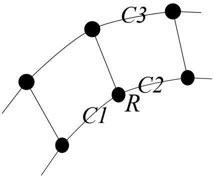

Figure: State P_0

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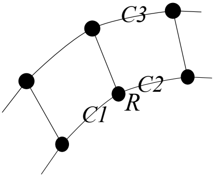
Robber only has one option.



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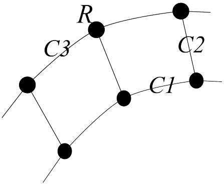
The cops then move to their endgame configuration:



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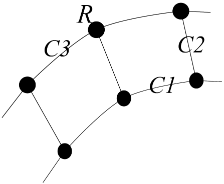

Cops win on their next turn regardless of robber's move. □

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Is $G \square K_2$ containable when G is containable?

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No.

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Is $G \square K_2$ containable when G is containable?
No. Counterexample: hypercubes.

Hypercubes

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Proof.
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Proposition

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In fact, at least $2n-2$ cops are required.

Hypercubes are not containable: Proof.

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Four cases:

1. 0 cops incident with robber.
2. Exactly 1 cop incident with robber.
3. Exactly k cops incident with robber ($1 < k < n-1$).
4. Exactly $n-1$ cops incident with robber.

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WLOG cop is on edge $\{v, v_n\}$. Every other cop can be adjacent to at most 2 of the vertices in $\{v_1, \ldots, v_{n-1}\}$ on the next cop move.

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Case 4: Exactly $n-1$ cops incident
An additional $n-1$ cops must be incident with robber's one escape vertex. □

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Hypercubes, continued

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We can prove something stronger if we think about retracts.

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The analogous result holds for Cops & Robber, too.

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Examples.

- Can retract K_3 onto K_2, but not cubically
- Can retract C_4 onto K_2 either as a cubical retract or not (either send both vertices outside the subgraph onto different vertices or the same vertex of K_2)
- Q_{n+1} retracts cubically onto $Q^n \times \{0\} \cong Q^n$ by setting the last coordinate to 0.

Cubical retracts

Theorem

Let $H \subset G$ be a cubical retract of G under ϕ. Then

$$\xi(G) \leq \max\{\xi(H), \xi(G-H)\} + dd(G,H) + \Delta(H) - 1$$

where $dd(G,H) = \max_{x \in H}(d_G(v) - d_H(v))$ is the **degree discrepancy** of H.

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where $dd(G, H) = \max_{x \in H}(d_G(v) - d_H(v))$ is the **degree discrepancy** of H.

Lemma

Suppose that we are playing a containment game on a graph G and that there are at least $c(G) + k - 1$ non-tail cops, then k new tail cops can be attached to R.

Cubical retracts

Proof.
Let

$$m = dd(G, H) + \Delta(H) + c(H) - 2$$

and

$$n = \max\{\xi(H), \xi(G - H)\} - c(H) + 1.$$

So we're showing $\xi(G) \leq m + n$.

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So we're showing $\xi(G) \leq m + n$.
Start with $m + n$ cops.
Phase 1: we use m of the cops to attach $\Delta(H) + dd(G, H) - 1$ tails to $\phi(R)$ in H (by lemma).

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Proof.
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$$n = \max\{\xi(H), \xi(G - H)\} - c(H) + 1.$$

So we're showing $\xi(G) \leq m + n$.
Start with $m + n$ cops.
Phase 1: we use m of the cops to attach $\Delta(H) + dd(G, H) - 1$ tails to $\phi(R)$ in H (by lemma). Now there are $n + c(H) - 1 = \max\{\xi(H), \xi(G - H)\}$ non-tail cops left.

Cubical retracts

Proof, cont'd.
Phase 2: these cops move until either the robber leaves H or they contain him on H.

Cubical retracts

Proof, cont'd.
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Cubical retracts

Corollary

$\xi(Q_n) \leq \dfrac{n(n-1)}{2}$ *for all* $n \geq 3$.

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Proof.
$dd(Q_{n+1}, Q_n) = 1$ and $\Delta(Q_n) = n$, so

$$\xi(Q_{n+1}) \leq \xi(Q_n) + 1 + n - 1 = \xi(Q_n) + n.$$

Use $\xi(Q_3) = 3$ and induction to get the desired result. □

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If G has girth at least 7 and is Δ-regular ($\Delta > 2$), then G is not containable by $\Delta + 1$ cops.

More general result on containment number

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For all G, $c(G) \leq \xi(G) \leq \Delta(G)\gamma(G)$.

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Upper bound: place a cop on each of the edges incident with each of the vertices in a dominating set of G. They can capture the robber in one step. □

Containment number conjecture

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This conjecture does hold "on average" in many random graphs [8].
$c(Q_n) = \lceil \frac{n+1}{2} \rceil$ (see [6]), so hypercubes provide an infinite class of examples where $\xi(G)$ is strictly less than $\Delta(G)c(G)$.

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- What happens if the game is played on non-reflexive graphs? $\xi(T) = 1$ for all trees and the Petersen graph becomes containable. Non-reflexive containability should probably be defined as $\xi(G) = \delta(G)$.

Thank you!

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