Balanced Partitions

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Abstract A famous theorem of Euler asserts that there are as many partitions of n into distinct parts as there are partitions into odd parts. We begin by establishing a less well-known companion result, which states that both of these quantities are equal to the number of partitions of n into even parts along with exactly one triangular part. We then introduce the characteristic of a partition, which is determined in a simple way by the placement of odd parts within the list of all parts. This leads to a refinement of the aforementioned result in the form of a new type of partition identity involving characteristic, distinct parts, even parts, and triangular numbers. Our primary purpose is to present a bijective proof of the central instance of this new type of identity, which concerns balanced partitions—partitions in which odd parts occupy as many even as odd positions within the list of all parts. The bijection is accomplished by means of a construction that converts balanced partitions of $2n$ into unrestricted partitions of n via a pairing of the squares in the Young tableau.

Keywords Integer partition · Distinct parts · Even parts · Triangular number · Characteristic · Bijection

Mathematics Subject Classification (2000) MSC 05A17 · MSC 11P81

1 Introduction

There are as many partitions of n into distinct parts as odd parts; this result still retains its appeal centuries after Euler proved it in 1748 and has formed the basis for generalizations in many directions, some of which are outlined in [1], [2], [3] and [4], among others. One natural avenue of inquiry is to ask whether there is a corresponding relationship between partitions into distinct parts and partitions involving even parts. The answer is in the affirmative: there are as many partitions of n into distinct parts as there are partitions of n into even parts along with exactly one triangular part. (A triangular part has size $T_k = \frac{1}{2}k(k+1)$ for some integer k.)

Notwithstanding its elementary nature, this partition identity seems to have been overlooked in recent decades. However, other relationships between partitions into distinct parts and even parts have been found in [2], [5] and [6], for example. It is interesting to note that in each case triangular numbers make an appearance, either overtly or implicitly.

Our aim is not so much to establish this identity (there is a short generating function proof), but to highlight its existence and to show that the search for a bijective proof leads in fruitful directions. In particular, we will define a quantity called the characteristic of a partition and demonstrate how it affords a refinement of this identity, thus leading to a new type of partition identity equating partitions of n into distinct parts having characteristic k with partitions of $n - T_{2k}$ into even parts.

The chief purpose of this paper is to present a bijective proof of the central instance of this type of identity. One formulation of this result states that for all n the number of balanced partitions of $2n$ into distinct parts is the same as the number of unrestricted partitions of n. A balanced partition is one in which the odd parts are equally split between odd positions and even positions when the parts are listed as usual in nonincreasing order. Thus the five balanced partitions of 8 are 8, 7–1, 6–2, 5–3 and 4–3–1, which agrees with the fact that $p(4) = 5$.

2 An initial result

Recall that the k^{th} triangular number is given by $T_k = \frac{1}{2}k(k+1)$. A triangular part of a partition is a part whose size is equal to T_k for some integer k. Thus unlike an even part, a triangular part may have size zero. As usual, the empty partition counts as a partition of 0 into distinct parts or into even parts.

Proposition 1 *For every nonnegative integer n, the number of partitions of n into distinct parts is equal to the number of partitions of n into even parts along with precisely one triangular part.*

To clarify the assertion, consider the cases $n = 9$ and $n = 10$. The partitions of n of each type for these values are listed in Table 1. In the right-hand column of each list the triangular part is shown in boldface. Note that in the second list the partitions **0**–6–4 and **6**–4 are counted separately, since the triangular part is distinguished. As predicted, there are an equal number of each type of partition in each list. This fact is quickly established using the Jacobi triple product.

Proof By substituting $x = q^{\frac{1}{2}}$ and $y = q^{\frac{1}{4}}$ in the Jacobi triple product

$$\prod_{m=1}^{\infty}(1-x^{2m})(1+x^{2m-1}y^2)(1+x^{2m-1}y^{-2}) = \sum_{n=-\infty}^{\infty} x^{n^2}y^{2n}, \tag{1}$$

we obtain the identity

$$\prod_{m=1}^{\infty}(1-q^m)(1+q^m)(1+q^{m-1}) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2+\frac{1}{2}n}. \tag{2}$$

Table 1 Partitions of 9 and 10 into distinct parts or even parts and a triangular part.

distinct	$\chi(\pi)$	△ + evens
9	-1	**3**–6
8–1	1	**3**–4–2
7–2	-1	**3**–2–2–2
6–3	1	**1**–8
6–2–1	-1	**1**–6–2
5–4	-1	**1**–4–4
5–3–1	-1	**1**–4–2–2
4–3–2	1	**1**–2–2–2–2

distinct	$\chi(\pi)$	△ + evens
10	0	**10**
9–1	0	**6**–4
8–2	0	**6**–2–2
7–3	0	**0**–10
7–2–1	-2	**0**–8–2
6–4	0	**0**–6–4
6–3–1	0	**0**–6–2–2
5–4–1	-2	**0**–4–4-2
5–3–2	0	**0**–4–2–2–2
4–3–2–1	2	**0**–2–2–2–2–2

Let $T(q) = 1+q+q^3+q^6+q^{10}+\cdots$ be the generating function for a single triangular part. Dividing the above equality by 2 and reindexing the $(1+q^{m-1})$ term leads to

$$\prod_{m=1}^{\infty}(1-q^{2m})(1+q^m) = T(q), \tag{3}$$

which may be rewritten more productively as

$$(1+q)(1+q^2)(1+q^3)\cdots = T(q)\cdot\frac{1}{1-q^2}\cdot\frac{1}{1-q^4}\cdot\frac{1}{1-q^6}\cdots.$$

The coefficient of q^n on the left counts partitions of n into distinct parts, while the coefficient of q^n on the right tallies partitions of n into even parts and exactly one triangular part, so we are done. □

Let $p(n)$ denote the number of unrestricted partitions of n and let $p_d(n)$ be the number of partitions of n into distinct parts. Since a partition of $2n$ into even parts is equivalent to a partition of n, there will be $p(\frac{1}{2}(9-1)) = p(4) = 5$ partitions of 9 involving a triangular part of 1 and even parts otherwise. Similarly, there will be $p(\frac{1}{2}(9-3)) = p(3) = 3$ partitions that employ a triangular part of 3, as evidenced by Table 1. Hence we deduce that $p_d(9) = p(4)+p(3)$. In general, this line of reasoning leads to an expression for $p_d(n)$ reminiscent of Euler's Pentagonal Number Theorem. Adopting the standard convention that $p(n) = 0$ for values of n other than nonnegative integers, we have the following.

Corollary 1

$$p_d(n) = \sum_{k=0}^{\infty} p\left(\tfrac{1}{2}(n-T_k)\right). \tag{4}$$

It is interesting to note the similarity of this formula with a relatively recent result obtained by Robbins [6], which states that

$$p_2(n) = \sum_{k=0}^{\infty} p(n-T_k), \tag{5}$$

where $p_2(n)$ is the number of partitions of n in two colors into distinct parts.

3 Characteristic of a partition

We now consider how a bijective proof of Proposition 1 might be obtained. The partitions of n into even parts and a triangular part are naturally grouped by the triangular part used, so we begin by searching for some feature of the partitions of n into distinct parts that gives rise to groups of the same sizes. After some searching we discover that the characteristic of a partition has the desired property.

Definition 1 Let π be a partition of n, with parts listed in nonincreasing order as usual. Let a_π be the number of odd parts appearing in even positions within the list, and let b_π be the number of odd parts appearing in odd positions. We define the *characteristic* $\chi(\pi)$ of the partition π to be the quantity $a_\pi - b_\pi$. When $\chi(\pi) = 0$, meaning that $a_\pi = b_\pi$, we say that the partition is *balanced*.

This definition applies to any partition of n, not necessarily into distinct parts. Also, we declare that $\chi(\pi) = 0$ for the empty partition.

To illustrate, let π be the partition 7–2–1. Since the odd parts occur in the first and third positions, we have $a_\pi = 0$ and $b_\pi = 2$, so $\chi(\pi) = -2$. Consulting Table 1 we find that among the partitions of 10 into distinct parts, $\chi(\pi) = -2$ also for 5–4–1, while $\chi(\pi) = 2$ only for 4–3–2–1; all other partitions satisfy $\chi(\pi) = 0$. Apparently partitions with $\chi(\pi) = 0$, -2 and 2 should correspond to partitions involving a triangular part of 0, 6 and 10, respectively. Examining the list for $n = 9$ further suggests that partitions with $\chi(\pi) = -1$ or 1 should pair off with partitions having a triangular part of 1 or 3, respectively. In general, we propose the following.

Conjecture 1 Let n be a fixed nonnegative integer. For each integer k, there are as many partitions of n into distinct parts having characteristic k as there are partitions of n into even parts and a single triangular part equal to T_{2k}. That is to say, there are $p\left(\frac{1}{2}(n - T_{2k})\right)$ such partitions.

Therefore we have found a refinement of the initial result outlined in Proposition 1; summing over all integers k produces Corollary 1. In the subsequent sections we will provide a bijective proof of this conjecture in the case of balanced partitions.

Remark 1 Based on this conjecture one can determine the generating function for $p_d(n,k)$, the number of partitions of n into distinct parts having characteristic k. Relatively elementary manipulations reveal that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} p_d(n,k)x^n y^k &= P(x^2)\sum_{k=-\infty}^{\infty} y^k x^{T_{2k}} \\ &= \prod_{m=1}^{\infty}(1+x^{2m})(1+x^{4m-1}y)(1+x^{4m-3}y^{-1}), \qquad (6)\end{aligned}$$

where $P(x) = \prod_{m=1}^{\infty}(1-x^m)^{-1}$ is the generating function for unrestricted partitions. The first equality is obtained by considering even and odd values of k separately, while the second follows from a routine application of the Jacobi triple product. Observe that specializing to $y = 1$ neatly gives $P_d(x)$, the generating function for partitions of n into distinct parts. It would be interesting to ascertain the generating function for $p(n,k)$, the number of partitions of n having characteristic k.

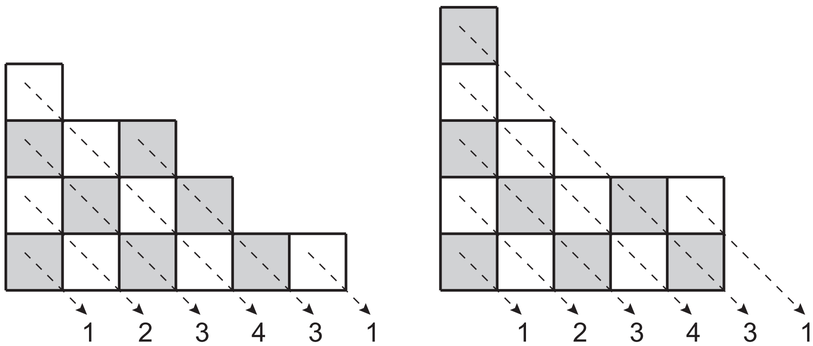

Fig. 1 Determining diagonal lengths from a Young tableau.

The characteristic of a partition of n may also be computed via the lengths of certain diagonals in its Young tableau. Let the lengths of successive diagonals (slanting from upper left to lower right) be denoted by d_1 through d_n, beginning with the lower left corner, as depicted in Figure 1. Note that squares within the same diagonal need not be adjacent. (For our purposes it will be more natural to order rows of the Young tableau in ascending order, "French style.") Thus the partition 6–4–3–1 has diagonals of length $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = 4$, $d_5 = 3$, $d_6 = 1$, and $d_k = 0$ for $7 \leq k \leq 14$, as shown. Observe that the partition 5–5–2–1–1 yields precisely the same values for d_1 through d_{14}. Thus different partitions may have the same diagonal lengths.

We mention without proof that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers are the diagonal lengths for some partition of n if and only if the following three conditions are met:

i. $d_1 + d_2 + \cdots + d_n = n$,
ii. $d_1 = 1, d_2 = 2, \ldots, d_m = m$ for some $1 \leq m \leq n$, and
iii. $d_m \geq d_{m+1} \geq \cdots \geq d_n$.

We will also require the following result on diagonal lengths.

Proposition 2 *Let π be a partition of n having diagonal lengths $d_1, d_2, \ldots, d_n$. Then*

$$\chi(\pi) = \sum_{k=1}^{n} (-1)^k d_k. \tag{7}$$

Furthermore, there is exactly one partition of n into distinct parts having the given diagonal lengths.

Proof Color the Young tableau for π in a chessboard fashion so that the lower left corner is shaded, as done in Figure 1. Then the odd-numbered diagonals will contain dark squares while the even-numbered diagonals contain light squares, so the sum $\sum(-1)^k d_k$ measures the signed excess of light squares. On the other hand, an even part yields a row of even length, which will contain an equal number of light and dark squares. Meanwhile an odd part in an even (resp. odd) position corresponds to a row with one extra light (resp. dark) square. Therefore $\chi(\pi) = a_\pi - b_\pi$ also measures the signed excess of light squares, so these quantities are equal.

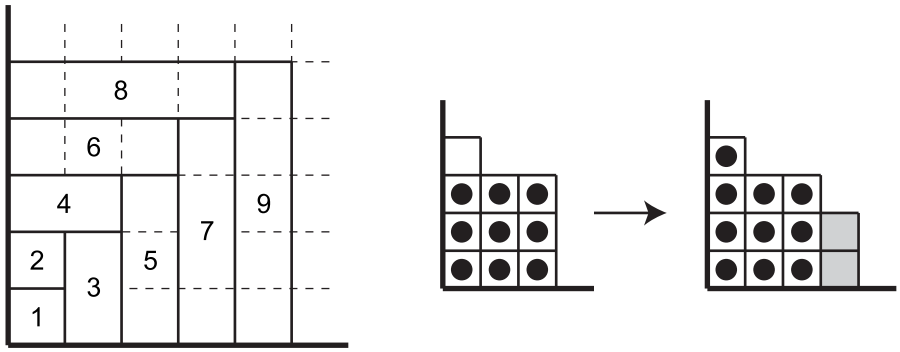

Fig. 2 Blocks used for creating a partition of n, numbered according to the diagonal they serve.

Finally, a Young tableau represents a partition into distinct parts if and only if all the squares within each diagonal are adjacent to one another and extend to the lower edge. Otherwise examine the Young tableau at a point along the lowest numbered diagonal where a break occurs to find a pair of equal parts. Hence there exists a unique partition of n into distinct parts having a particular set of diagonal lengths, obtained from a given tableau by "sliding" all blocks within each diagonal down and to the right as far as possible. □

It is worth noting that one may define an equivalence relation on the set of all partitions of n by declaring that $\pi_1 \sim \pi_2$ whenever π_1 and π_2 have the same diagonal lengths. Then Proposition 2 indicates that there is exactly one partition of n into distinct parts within each equivalence class, hence there are $p_d(n)$ classes in total. In addition, the characteristic of a class is well-defined. (*N.B.* The fact that $\chi(\pi)$ is given by an alternating sum explains our choice of terminology for this quantity.)

4 Describing the bijection

For the remainder of our discussion we will focus solely on balanced partitions, with the goal of proving that the number of balanced partitions of $2n$ into distinct parts is equal to the number of partitions of n. Consider the Young tableau for any balanced partition of $2n$ having diagonal lengths d_1 to d_{2n}, shaded as in Figure 1. We perform the following algorithm, which has the effect of pairing light and dark squares in the process of creating a partition of n.

a. Keep the dark square from the first diagonal in its original position.
b. For each subsequent diagonal, use as many squares as necessary from that diagonal to cover all unpaired squares (if any) left over from the previous step.
c. Place the remaining squares in the rectangular block reserved for that particular diagonal as indicated in Figure 2, starting at the left-hand (or bottom) edge and filling in to the right (or up).
d. Repeat steps b. and c. until all diagonals have been incorporated and every square has been paired with a square of the opposite color.

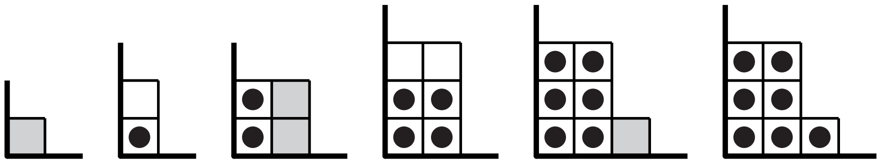

Fig. 3 Applying the bijective algorithm to a partition with diagonal lengths 1, 2, 3, 4, 3 and 1.

One step of this algorithm is illustrated in Figure 2 for a certain partition with $d_7 = 3$. Squares along the seventh diagonal are dark; we use one of them to cover the currently unpaired white square in the fourth row, then begin filling the block in the fourth column with the remaining two dark squares. (The solid circles represent squares that have already been paired.) The entire process is illustrated in Figure 3 for the partitions in Figure 1, ultimately yielding the partition 3–2–2.

Before continuing, the reader is encouraged to perform this algorithm for each of the balanced partitions of 10 into distinct parts. (They are 10, 9–1, 8–2, 7–3, 6–4, 6–3–1 and 5–3–2.) This can easily be done with five red and five black cards from a standard deck to represent the five light and dark squares. It is quite marvelous to see the Young tableaux for each partition of 5 appear in turn. The intuition gained from such an exercise will also greatly clarify the subsequent arguments.

Proposition 3 *Let π be a balanced partition of $2n$. Then the algorithm described above yields a valid Young tableau for a partition of n.*

Proof Suppose π has diagonal lengths $d_1 = 1$, $d_2 = 2$, $\ldots$, $d_m = m$ followed by lengths satisfying $d_m \geq d_{m+1} \geq \cdots \geq d_{2n}$. (Here m is a constant depending on π.) The first m steps of the algorithm proceed in an orderly fashion: the squares within each diagonal cover all unpaired squares from the previous step and then exactly fill out their designated block shown in Figure 2. During this stage the Durfee square of the new partition is filled out.

We will show that from this point on each diagonal is long enough to cover all the unpaired squares left over from the previous step, but not so long as to subsequently fill the allotted space in the block reserved for that diagonal. Assume for sake of argument that we are handling diagonal $2k$, which has light squares, for some $2k > m$. Then there should be at least as many light as dark squares in the first $2k$ diagonals, but the excess should be less than k. In other words, we must have

$$0 \leq (d_{2k} + d_{2k-2} + \cdots + d_2) - (d_{2k-1} + d_{2k-3} + \cdots + d_1) < k.$$

But $d_{2j} - d_{2j-1} = 1$ when $2 \leq 2j \leq m$ and $d_{2j} - d_{2j-1} \leq 0$ for $2j > m$, which establishes the right-hand inequality. Furthermore, using the fact that $\chi(\pi) = 0$ we may rewrite the left-hand inequality as $(d_{2n} + \cdots + d_{2k+2}) - (d_{2n-1} + \cdots + d_{2k+1}) \leq 0$, which follows immediately from the fact that $d_{2j} - d_{2j-1} \leq 0$ whenever $2j > m$.

For diagonal $2k+1 > m$ consisting of dark squares we must instead show that

$$0 \leq (d_{2k+1} + d_{2k-1} + \cdots + d_1) - (d_{2k} + d_{2k-2} + \cdots + d_2) < k+1.$$

Pairing terms and using $d_1 = 1$ gives the right-hand inequality in the same manner as above. We may use $\chi(\pi) = 0$ to rewrite the left-hand inequality as before, then pair up terms and note that $d_{2n} \geq 0$ to finish the odd case.

To complete the proof we must show that the portions of the Young tableau to the right of and above the Durfee square form a nonincreasing sequence of columns and rows. The height of the $(k+1)^{\text{st}}$ column is equal to the excess of dark squares in the first $2k+1$ diagonals, thus is given by

$$(d_{2k+1} + d_{2k-1} + \cdots + d_1) - (d_{2k} + d_{2k-2} + \cdots + d_2).$$

Hence the difference in height between the k^{th} and $(k+1)^{\text{st}}$ columns is $d_{2k} - d_{2k+1}$, which is nonnegative since $2k+1 > m$. (I.e. we are to the right of the Durfee square.) The same reasoning shows that the rows above the Durfee square are also nonincreasing, thus completing the proof. □

5 Proof of the bijection

We now show that the construction just described is in fact a bijection, which will prove our main result.

Theorem 1 *For each nonnegative integer n, the number of balanced partitions of $2n$ into distinct parts is equal to the number of partitions of n.*

Proof When $n = 0$ there is one partition of each type, so assume that $n \geq 1$. There are $p_d(2n)$ equivalence classes of partitions of $2n$ when they are grouped according to diagonal lengths, since there is exactly one partition into distinct parts within each class, by Proposition 2. The above construction maps each class to a partition of n, so we must establish that this map is injective and surjective to prove the theorem.

Suppose that partitions π_1 and π_2 belong to distinct classes, and let their diagonal lengths differ for the first time at diagonal k. Then clearly the construction will result in a different number of filled squares appearing in block k in Figure 2. Since no later step in the construction affects the number of squares in that block, the resulting partitions of n will be distinct, hence the map is injective.

We now show that every partition of n arises via our construction. Given a Ferrer's diagram for a partition of n, overlay the blocks used in the algorithm, as illustrated in Figure 4 for the partition 6–5–4–2–2. Set $d_1 = 1$, and for each $1 < j \leq 2n$ define d_j to be the total number of dots contained within blocks j and $j-1$. We claim that these values constitute a valid set of diagonal lengths for a partition of $2n$. For instance, the nonzero values of the d_j for the partition illustrated in Figure 4 are d_1 through d_{12}, equal to 1, 2, 3, 4, 5, 5, 5, 5, 4, 2, 1, 1. These correspond to the balanced partition 12–9–7–6–4 of 38.

Note that all n dots are contained within the $n \times (n+1)$ rectangle consisting of blocks 1 to $2n$. Also, every dot is counted exactly twice when assigning values for the diagonal lengths (since no dots reach block $2n$), so $d_1 + d_2 + \cdots + d_{2n} = 2n$. Next let m be the smallest value for which block m is completely filled but block $m+1$ is not. Then clearly we have $d_1 = 1$, $d_2 = 2$, …, $d_m = m$. But as soon as the dots fail

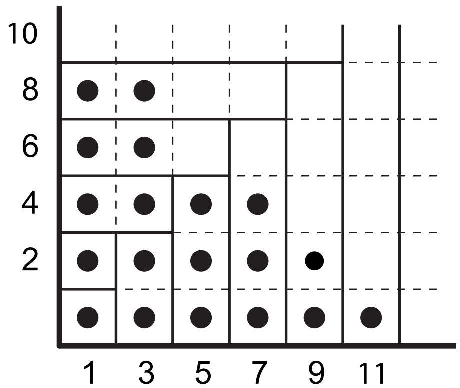

Fig. 4 Proving that the construction is surjective in the case of a 6–5–4–2–2 partition.

to completely fill one of the blocks—as is the case for block 6 in Figure 4—the dots in all subsequent blocks may not extend beyond those in the previous adjacent block, since this would imply that some row or column of the Ferrer's diagram had a gap. Note also that block $m+1$ has no more dots than block $m-1$, since the latter block is filled but the former is not. By definition $d_k - d_{k+1}$ is equal to the difference between the number of dots in (adjacent) blocks $k-1$ and $k+1$, which we have just argued is zero or positive when $k \geq m$. Therefore we conclude that $d_m \geq d_{m+1} \geq \cdots \geq d_{2n}$.

In summary, we have shown that the values for d_1 through d_{2n} represent the diagonal lengths for some partition of $2n$. Hence the map defined by our construction is surjective, and therefore bijective, which completes the proof. □

Corollary 2 *The number of ordered $2n$-tuples $(d_1, d_2, \ldots, d_{2n})$ of nonnegative integers satisfying the following conditions is equal to $p(n)$.*

i. $d_1 + d_2 + \cdots + d_{2n} = 2n$,
ii. $d_1 + d_3 + \cdots + d_{2n-1} = d_2 + d_4 + \cdots + d_{2n}$,
iii. $d_1 = 1$, $d_2 = 2, \ldots, d_m = m$ *for some* $1 \leq m \leq 2n$,
iv. $d_m \geq d_{m+1} \geq \cdots \geq d_{2n}$.

Proof Such ordered $2n$-tuples comprise all the possible diagonal lengths for balanced partitions of $2n$, which are in one-to-one correspondence with partitions of n, by Theorem 5. □

Remark 2 We made a choice to orient even-numbered blocks horizontally and odd-numbered blocks vertically when setting up our algorithm. However, the mechanics of the proofs would have proceeded just as smoothly if Figure 2 were reflected over the line $y = x$. In this case the algorithm would produce a partition conjugate to the one it currently produces, as one might imagine. We chose the placement of blocks described above because this version of the algorithm has the additional property that it converts a balanced partition of $2n$ with all even parts to the partition of n having parts half as large, as the reader may confirm.

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