$\square$

## Set Theory

### 2.1 Presenting Sets

Certain notions which we all take for granted are harder to define precisely than one might expect. In Taming the Infinite: The Story of Mathematics, Ian Stewart describes the situation in this way:

The meaning of 'number' is a surprisingly difficult conceptual and philosophical problem. It is made all the more frustrating by the fact that we all know perfectly well how to use numbers. We know how they behave, but not what they are.

He goes on to outline Gottlob Frege's approach to putting the whole numbers on a firm footing. Thus one might define the concept of 'two' via the collection of all sets containing two objects. However, this practice of considering all sets satisfying a certain condition cannot be applied indiscriminately, as philosophermathematician Bertrand Russell subsequently pointed out. In the Mathematical Outing on the next page you will consider "Russell's paradox," which highlights the potential problems with Frege's approach.

We will be content with a relatively informal definition of a set.

A set is any unordered collection of distinct objects. These objects are called the elements or members of the set. The set containing no elements is known as the empty set.

A set may have finitely many elements, such as the set of desks in a classroom; or infinitely many members, such as the set of positive integers; or possibly no elements at all. The members of a set can be practically any objects imaginable, as long as they are clearly defined. Thus a set might contain numbers, letters,

## Mathematical Outing $\star \star \star$

To obtain a sense of the sorts of pitfalls awaiting set theorists, consider the following classic paradox. In a certain town there is a single barber, who shaves exactly those men who do not shave
 themselves. Who shaves the barber? Try to appreciate the logical paradox that arises in this description of the barber, then find the clever trick answer that circumvents the paradox.

In a similar manner we could define a set which contains exactly those sets which do not contain themselves. Why does this lead to a logical inconsistency? And how is it possible for a set to contain itself in the first place?
polynomials, points, colors, or even other sets. In theory a set could contain any combination of these objects, but in practice we tend to only consider sets whose elements are related to one another in some way, such as the set of letters in your name, or the set of even numbers. As in the previous chapter, we will confine ourselves mainly to mathematical objects and examples.

We typically name sets using upper case letters, such as $A, B$ or $C$. There are a variety of ways to describe the elements of a set, each of which has advantages. We could give a verbal description of a set, for example, by declaring that $B$ is the set of letters in the title of this book. We might also simply list the elements of a set within curly brackets:

$$
B=\{b, r, i, d, g, e, t, o, h, i, g, h, e, r, m, a, t, h\}
$$

Recall that a set only catalogs distinct objects, so the appearance of the second letter $g$ is redundant and should be omitted, and similarly for other repeated letters. An equivalent but more appropriate list of the letters in this set is

$$
B=\{b, r, i, d, g, e, t, o, h, a, m\} .
$$

Since the order in which elements is listed is irrelevant, we could also write

$$
B=\{a, b, d, e, g, h, i, m, o, r, t\} \quad \text { or } \quad B=\{m, o, t, h, b, r, i, g, a, d, e\} .
$$

For a given set, it is natural to ask which objects are included in the set and how many objects there are in total. We indicate membership in or exclusion from a set using the symbols $\in$ and $\notin$. Thus it would be fair to say that $a \in B$ and $g \in B$, but $z \notin B$ and $\star \notin B$ either. We also write $|B|$ to indicate the size, or cardinality of set $B$. In the example above we have $|B|=11$, of course. For the time being we will only consider the cardinality of finite sets.


b) Think of a five letter word with the property that $|B|=3$, where $B$ is the set of letters appearing in your word.

Listing the elements of a set has its drawbacks when the set contains infinitely many members．However，when the pattern is clear it is acceptable to list the first four or five elements，followed by an ellipsis（．．．）．Thus the set of all positive odd integers is $\{1,3,5,7, \ldots\}$ ．The empty set can also be written using curly brackets as $\}$ ．However，this special set arises so frequently that it has been assigned its own symbol，which is $\emptyset$ ．
 3 or 4．List the elements of $C$ using an ellipsis．

Another standard method for presenting a set is to provide a mathematical characterization of the elements in the set．Suppose we wish to refer to the set of all real numbers greater than 5 ．The following notation，which we shall call bar notation for lack of a more imaginative term，achieves this quite efficiently． Using the symbol $\mathbb{R}$ for the set of all real numbers，we could write

$$
A=\{x \mid x \in \mathbb{R} \text { and } x>5\} .
$$

The actual element（in this case a number $x$ ）is placed to the left of the bar， while the description（ $x$ is a real number greater than 5）appears to the right of the bar．A literal translation would be＂$A$ is the set of all $x$ such that $x$ is a real number and $x$ is greater than 5．＂A less clunky rendition might read＂Let $A$ be the set of all real numbers greater than 5 ．＂Observe that $x$ is only used internally in the definition of set $A$ ；it does not refer to anything beyond．Thus we would obtain the same result by defining $A=\{w \mid w \in \mathbb{R}$ and $w>5\}$ ．
（品哊即T d）Let $A$ be the set of all real numbers between 5 and 6 ，including 5 but not 6 ．Describe $A$ with bar notation，using the variable $y$ ．

There is often more than one way to employ bar notation to describe a set． For example，suppose that $B$ is the set of perfect squares．We could think of the elements of $B$ as numbers，each of which is the square of an integer，in which case we would write $B=\left\{n \mid n=k^{2}, k \in \mathbb{Z}\right\}$ ．Or we might decide that the elements of $B$ are squares，such that the number being squared is an integer． This interpretation leads to $B=\left\{n^{2} \mid n \in \mathbb{Z}\right\}$ ．The latter approach is preferable in many ways，but either is correct．
 using bar notation．

Certain sets of numbers，such as the real numbers $\mathbb{R}$ ，are referred to regularly enough to merit their own special symbol．Other standard sets include the integers $\mathbb{Z}$ ，the positive integers $\mathbb{N}$（also called the natural numbers），the rational numbers $\mathbb{Q}$（the set of all fractions），and the complex numbers $\mathbb{C}$ ．Recall that a complex number is formed by adding a real number to a real multiple of $i$ ， where $i=\sqrt{-1}$ ．Thus we could write

$$
\mathbb{C}=\{a+b i \mid a, b \in \mathbb{R}\} .
$$

In case you were wondering，the letter $\mathbb{Z}$ for＇integers＇comes to us compliments of the German root word zahl，meaning＇number．＇

日筑SWERS
a）Set $A$ must be the empty set：$A=\emptyset$ ．
b）Possible answers include radar，geese，queue or mommy．
c）$C=\{3,4,33,34,43,44,333,334, \ldots\}$ ．It is probably a good idea to list at least seven elements of this set before the ellipsis，to make the pattern clear． d）We could write $A=\{y \mid y \in \mathbb{R}, 5 \leq y<6\}$ ．
e）Either $C=\{n \mid n=2 k+1, k \in \mathbb{Z}\}$ or $C=\{2 n+1 \mid n \in \mathbb{N}\}$ will work，although the latter is more concise．
興 It appears at first that the question cannot be answered．If the barber shaves himself，then he is shaving a man who shaves himself，contrary to his job description． On the other hand，if he does not shave himself，then he neglects his mandate to shave all men who don＇t shave themselves．The way out of this quandary，of course，is to realize that the barber is a woman！

There is no similar clever fix for Russell＇s paradox，though．Our set $S$ contains itself or it doesn＇t；either situation contradicts the definition of the set．However，it is possible for a set to contain itself．For instance，let $A=\{1,\{1,\{1,\{1, \ldots\}\}\}\}$ ．Then $A=\{1, A\}$ ．There are infinitely many nested sets in this example，although each set contains only two elements－the number 1 and the set $A$ ．

## ExERCISES

1．What is the set of colors appearing on both the American flag and the Ja－ maican flag？

2．Give a verbal description of the set $\{1,4,8,9,16,25,27,32,36, \ldots\}$ ．（In other words，find the rule that determines which numbers are included on this list．）

3．Give a verbal description of the set \｛January，March，May，July，August， October，December\}.

4．Give an example of a set $B$ for which $|B|=3$ and the elements of $B$ are polynomials having even coefficients．
5．Give an example of a set $C$ with $|C|=2$ such that the elements of $C$ are sets each of which contain four letters．

6 ．Let $D$ be the set whose elements are equal to the product of two consecutive natural numbers，such as $12=3 \cdot 4$ ．Present set $D$ using a list and also via bar notation．Which method is better suited for this set？

7．Briefly justify why the following statements are true or false．
a）If $A$ is the set of letters in the word＇flabbergasted，＇then $|A|=13$ ．
b）For the set $A$ in the previous part，we have $a \in A$ or $z \in A$ ．
c）If $B=\{n \mid n \in \mathbb{Z}, 10 \leq n \leq 20\}$ then $|B|=10$ ．
d）For the set $B$ in the previous part we have $11 \in B$ and $\sqrt{200} \in B$ ．
e）If $L$ is the set of letters in your full legal name，then $a \in L$ ．
f）Let $C=\left\{x \mid x \in \mathbb{R}, x^{2} \leq 10\right\}$ ．If $\pi \notin C$ then $-3 \notin C$ ．
8．How many sets $A$ are there for which $|A|=5$ and the elements of $A$ are states in New England？
9. Describe the following sets using bar notation.
a) $A$ is the set of all integers divisible by 7
b) $B=\{2,3,5,9,17,33,65, \cdots\}$
c) $C$ is the set of all real numbers between $\sqrt{2}$ and $\pi$
d) $D=\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$
10. Using bar notation, describe the set of rational numbers between 0 and 1 . Then describe the set of positive rationals whose denominator is a power of 2 , such as $\frac{7}{2}, \frac{3}{4}, 5$ or $\frac{1}{16}$. (The powers of 2 are $1,2,4,8, \ldots$ )
11. Consider the set $\{y=m(x-1) \mid m \in \mathbb{R}\}$. Give a verbal description of the sorts of objects that are elements of this set.
12. Let $B=\{2 m+5 n \mid m, n \in \mathbb{N}\}$. Is $10 \in B$ ? Is $13 \in B$ ? Explain.

## Writing

13. Let $A$ be a set with $|A| \geq 3$, all of whose elements are integers. Show that one can find distinct elements $m, n \in A$ such that $m-n$ is even.
14. Let $A, B$ and $C$ be different sets containing letters of the alphabet. Explain why there must exist some letter that is either contained in exactly one of the sets or contained in exactly two of the sets.
15. Prove that it is impossible to split the natural numbers into sets $A$ and $B$ such that for distinct elements $m, n \in A$ we have $m+n \in B$ and vice-versa.
16. Set $C$ consists of the thirty-six points of the form $(a, b)$ where $a$ and $b$ are integers with $0 \leq a, b \leq 5$. Prove that no matter how we select five points from set $C$, two of them will be situated a distance of $2 \sqrt{2}$ or less apart.

### 2.2 Combining Sets

Membership in the exclusive $\Delta \Pi$ club is not for everyone. Only those people whose first and last names both begin with the letter D and whose birthday is $3 / 14$ are permitted to join. In other words, the $\Delta \Pi$ club is only interested in those rare individuals common to both categories. Lately the club president, Daphney Daly, has suggested that in order to boost the club's dwindling enrollment, membership restrictions should be relaxed to allow individuals in either category to apply. These two approaches to membership requirements correspond in a natural way to the two most basic means of combining sets.

The set of elements common to two given sets $A$ and $B$ is known as their intersection and written as $A \cap B$. The set of elements appearing in at least one of these sets is called the union, denoted by $A \cup B$.

口ப热ץ a) Decide which elements ought to belong to each of $A \cup B \cup C$ and $A \cap B \cap C$. Then write a compact description of each set using bar notation.

## Mathematical Outing $\star \star \star$

Imagine that a certain math class consists of both male and female students，some of whom reside in New York while others come from out of state．All students are currently seated．You are permitted
 to request that all the students within some broad category（boys，girls，in－state or out－of－state）stand up．You may also ask all male students，or all female students，to reverse their position by standing if they are currently seated or vice－versa．However，you may not give instructions such as＂All boys please sit，＂or＂All girls from out of state please stand．＂Figure out how to arrange for the following sets of students to stand while all others are seated．
－All students who are either female or from out of state．
－All female students who are from New York．
－All students who are either female and from New York or male and from out of state．

Note that the set operation of intersection corresponds to the logical opera－ tion of conjunction．This relationship is made clear by the fact that

$$
A \cap B=\{x \mid x \in A \text { and } x \in B\} .
$$

Similarly union corresponds to the logical operation of disjunction，since

$$
A \cup B=\{x \mid x \in A \text { or } x \in B\} .
$$

Notice the resemblance between the symbols $\cap, \wedge$ and $\cup, \vee$ as well．
工吅工些 E ）Suppose that $x \in A \cup B \cup C$ but $x \notin A \cap B \cap C$ ．Consequently， how many of the sets $A, B$ or $C$ must $x$ be an element of？

It would stand to reason that the set operation corresponding to NOT would involve creating a new set consisting of all objects not contained in a set $A$ ．Some care needs to be exercised here，though．For instance，if $A$ is the set of students registered for our course who are sophomores，then objects not contained in $A$ include the governor of Maine，the color orange，and a golden retriever named Izzy，among many other things．What we really have in mind when we imagine ＂not $A$＂is the set of all students who are registered for our course who are not sophomores．There is a universal set lurking in the background that indicates the set of all objects under consideration；in our case，students registered for this course．Working within a universal set also helps
 to dodge the paradoxes implicit in dealing with＂the set of all sets．＂

With this in mind, let $U$ be a universal set, and let $A$ be a set whose elements all belong to $U$. Then the complement of set $A$, denoted by $\bar{A}$, is comprised of all elements of $U$ which are not in $A$. (The set $\bar{A}$ is sometimes referred to as the complement of $A$ in $U$.) The universal set is often understood from context. Thus if $B$ is the set of real numbers less than 5 , then $\bar{B}=\{x \mid x \in \mathbb{R}, x \geq 5\}$. It would be almost redundant to declare that $U$ is the set of real numbers.
 complement of $A$ ?
 $B=\{w, y, o, m, i, n, g\}$. Compute $|A \cup B|$ and $|A \cap \bar{B}|$.
It is also standard practice to omit any reference to the universal set when discussing statements such as $\overline{A \cap B}=\bar{A} \cup \bar{B}$. It is assumed that the elements of $A$ and $B$ belong to a larger universal set in which all the action takes place.

The stage is almost set for our first major set theoretic result. However, before attempting it we need a strategy for showing that two sets are equal.

We say that $A$ and $B$ are equal sets, written $A=B$, if these two sets contain precisely the same elements. One common technique for showing that two sets are equal is to show that every element of the first set must be an element of the second set, and vice-versa.

We employ this strategy to establish the set identity $\overline{A \cap B}=\bar{A} \cup \bar{B}$. Step one: Let $x$ be any element of the first set; i.e. let $x \in \overline{A \cap B}$. This means that $x \notin A \cap B$. Since $A \cap B$ consists of elements in both $A$ and $B$, if $x$ is not in the intersection then either $x \notin A$ or $x \notin B$, or both. In other words, $x \in \bar{A}$ or $x \in \bar{B}$, which means that $x \in \bar{A} \cup \bar{B}$.
Step two: On the other hand, if $x \in \bar{A} \cup \bar{B}$ then we know that $x \in \bar{A}$ or $x \in \bar{B}$, which means that $x \notin A$ or $x \notin B$. Since $x$ is missing from at least one of the sets $A$ or $B$, it cannot reside in their intersection, hence $x \notin A \cap B$. Finally, this is the same as $x \in \overline{A \cap B}$. Hence we conclude that the sets $\overline{A \cap B}$ and $\bar{A} \cup \bar{B}$ are indeed equal. ${ }^{\dagger}$

You may have noticed that the steps in this paragraph were essentially the same as the steps in the previous paragraph, just in the opposite order. This will sometimes be the case, but more often it will not, especially as we tackle more sophisticated set identities.

There is a rather convenient means of picturing unions, intersections and complements of sets which greatly clarifies set identities such as $\overline{A \cap B}=\bar{A} \cup \bar{B}$. A Venn diagram for two sets $A$ and $B$ is shown below. Given an arbitrary element $x$ of the universal set, there are four ways that $x$ could be (or not be) a member of set $A$ and set $B$. These possibilities correspond to the four regions in the Venn diagram. For example, we might have $x \notin A$ but $x \in B$,
which corresponds to region $I I I$ ．Various combinations of these regions represent different sets．Thus set $A$ is made up of regions $I$ and $I I$ ，while $A \cap B$ consists of region $I I$ alone．The remaining figures below illustrate how to shade in the portion of the Venn diagram corresponding to the sets $\overline{A \cap B}, \bar{A}$ and $\bar{B}$ ．It now becomes clear that $\bar{A} \cup \bar{B}$ will be identical to $\overline{A \cap B}$ ，so we conclude that these two sets are equal．


A Venn diagram for three sets is shown at right，with the region correspond－ ing to the set $(A \cup B) \cap C$ shaded．Because a Venn diagram for two or three sets includes regions for every possible combination of membership in the sets，they provide a rigorous means of confirming identities involving two or three sets．In other words，the pictures above（if presented in a more organized manner）serve to es－ tablish that $\overline{A \cap B}=\bar{A} \cup \bar{B}$ just as adequately as the two－paragraph proof that preceded them．For our purposes we will declare that the technique of Venn diagrams is valid as long as there are three or fewer sets involved，which are combined using
 only union，intersection，and complements．

（品唁即T f）Shade in the set $A \cup(B \cap C)$ in a Venn diagram for three sets． Compare it to the Venn diagram for $(A \cup B) \cap C$ above．What can you conclude based on these pictures？

a) We have that $A \cup B \cup C=\{x \mid x \in A$ or $x \in B$ or $x \in C\}$ and $A \cap B \cap C=\{x \mid x \in A$ and $x \in B$ and $x \in C\}$.
b) It is the case that $x$ belongs to exactly one or two of $A, B, C$.
c) The answer is the original set $A$.
d) We have $|A \cup B|=9$ and $|A \cap \bar{B}|=2$.
e) A three-set Venn diagram has eight regions.
f) The Venn diagram for $A \cup(B \cap C)$ will resemble the one pictured for $(A \cup B) \cap C$, except that the remaining regions within set $A$ will also be shaded in. Hence these two sets are not equal in general.
門•In the first scenario, simply have the female students rise, then ask the out of staters to rise. - For the second situation, ask the boys to stand, request that the out-of-state students also stand, then have all boys and all girls reverse positions. In the third case have the New Yorkers rise, then have all the boys reverse positions.

## ExERCISES

17. Let $A$ and $B$ be the sets of students in a certain class who are sophomores and who are from New York, respectively. Write an expression that represents the set of students who are sophomores or who come from outside New York.
18. Define a universal set $U=\{a, b, c, d, e, f, g, h\}$. Using these elements, construct two sets $A$ and $B$ satisfying $|A|=5,|B|=4$ and $|A \cap B|=2$. Using the sets you chose, compute $|\bar{A} \cap \bar{B}|$.
19. Why is it not possible for two sets to satisfy both $A \cap B=\{f, o, u, r\}$ and $A \cup B=\{f, o, r, t, y, s, i, x\}$ ?
20. Given the universal set $U=\{a, b, c, \ldots, z\}$, we define $A=\{b, r, i, d, g, e\}$, $B=\{f, o, r, t, y, s, i, x\}$ and $C=\{s, u, b, z, e, r, o\}$. Decide whether the following statements are true or false.
a) $|A \cup C|=10$
b) $B \cap \bar{B}=\emptyset$
c) $|B \cup \bar{C}|=23$
d) $(A \cup B) \cap C=\{s, o, b, e, r\}$
e) $|(A \cap B) \cup(B \cap C)|=5$
f) $\bar{A} \cap \bar{B} \cap \bar{C}=\{a, c, h, j, k, l, m, n, p, q, v, w, y\}$
g) $|A \cup B \cup C|=15$
21. Let $A=\{x \mid 1<x<3\}, B=\{x \mid 5 \leq x \leq 7\}$ and $C=\{x \mid 2<x<6\}$, where $x$ represents a real number. Determine the sets $A \cup C,(A \cup B) \cap C$ and $B \cap \bar{C}$, writing your answers in bar notation.
22. List the four possible ways that $x$ could be (or not be) an element of two given sets $A$ and $B$. In each case identify the corresponding region in the labelled Venn diagram within this section.
23. Use Venn diagrams to prove that $\overline{A \cap B \cap C}=\bar{A} \cup \bar{B} \cup \bar{C}$.
24. Use Venn diagrams to prove that $A \cap(B \cup C)=(A \cap B) \cup(A \cap C)$.

## Writing

25. Explain why our set equality strategy is valid. In other words, prove that if every element of a set $A$ is contained in another set $B$ and vice-versa, then the two sets must contain precisely the same elements.
26. Without appealing to a Venn diagram, demonstrate that $\overline{A \cup B}=\bar{A} \cap \bar{B}$.
27. Without appealing to a Venn diagram, prove that for any three sets $A, B$ and $C$ we have $A \cup(B \cap C)=(A \cup B) \cap(A \cup C)$.
28. Prove that $\overline{A_{1} \cap A_{2} \cap A_{3} \cap A_{4}}=\bar{A}_{1} \cup \bar{A}_{2} \cup \bar{A}_{3} \cup \bar{A}_{4}$.
29. Let $A$ and $B$ be sets within the universal set $U=\{a, b, c, \ldots, z\}$. Working from the definitions, explain why $|\overline{A \cup B}|=26-|A|-|B|+|A \cap B|$.
30. Prove that $|A \cup B|+|A \cup C|+|B \cup C| \leq|A|+|B|+|C|+|A \cup B \cup C|$ for any three finite sets $A, B$ and $C$.

## Further Exploration

31. Draw a configuration of four circles within a rectangle that creates as many regions as possible. Confirm that it is impossible to obtain the requisite sixteen regions necessary for a complete Venn diagram of four sets. Then figure out a way to create a Venn diagram for four sets using elliptical regions.

### 2.3 Subsets and Power Sets

We now introduce several concepts which concern the extent to which elements of one set are members of another set. At one extreme, it may be the case that all the elements of a set $A$ also belong to another set $B$. At the other extreme, it could also be the case that none of the elements of $A$ are contained in $B$.

Given sets $A$ and $B$, whenever each element of $A$ is also an element of $B$ we say that $A$ is a subset of $B$ and write $A \subseteq B$. Therefore to prove that $A \subseteq B$ one must show that if $x \in A$, then $x \in B$.

On the other hand, if $A$ and $B$ have no elements in common then they are disjoint, which can be proved by showing that if $x \in A$ then $x \notin B$.

It makes sense that if $A$ is a subset of $B$, then $B$ contains $A$. More formally, we say that $B$ is a superset of $A$, denoted by $B \supseteq A$. However, this perspective (and associated notation) arises fairly infrequently.
 of one another.
 we have $A \cap B=A$ ? if we have $A \cap B=B$ ?

## Mathematical Outing $\star \star \star$

According to the definition of subset，should the empty set be counted as a subset of $C_{2}=\{1,2\}$ ？ Decide whether the definition supports an answer of yes or no before going on．The other three
 subsets are clearly $\{1\},\{2\}$ and $\{1,2\}$ ，so depending upon your answer，you would conclude that there are either three or four subsets of $C_{2}$ ．

Now list all the subsets of $C_{1}=\{1\}, C_{3}=\{1,2,3\}$ and $C_{4}=\{1,2,3,4\}$ and look for a pattern among the numbers of such subsets．Which decision regarding the empty set leads to a nicer，more natural answer？Based on your pattern，how many subsets will $C_{n}=\{1,2, \ldots, n\}$ have？

To highlight the process by which we begin crafting a proof，let us show that if $A \subseteq B \cap \bar{C}$ then $C \subseteq \bar{A}$ ．There are two subsets here，so where do we begin？ The key is to focus on the statement to be proved；namely，$C \subseteq \bar{A}$ ．（This is the part following the word＇then＇in an if－then statement；i．e．the conclusion of the implication．）So we should apply our set inclusion strategy to $C \subseteq \bar{A}$ ：we begin by supposing that $x \in C$ and will attempt to prove that $x \in \bar{A}$ ．
口ப笈 inside $B \cap \bar{C}$ ．From the picture，is it feasible that $C \subseteq \bar{A}$ ？

Drawing on the intuition gained by the Venn diagram just constructed，we realize that since $x \in C$ it follows that $x \notin \bar{C}$ ．Hence by definition of intersection， $x \notin B \cap \bar{C}$ either．We can now make the key deduction in the proof：since $x$ is outside the set $B \cap \bar{C}$ but all of $A$ is contained within $B \cap \bar{C}$ ，we know that $x \notin A$ ．This means that $x \in \bar{A}$ ．Since $x \in C$ implies $x \in \bar{A}$ we may conclude that $C \subseteq \bar{A}$ ，as desired．${ }^{\dagger}$
（品䟚品 d）Can you touch your tongue and your nose？
A set is considered to be a subset of itself，so it is true that $A \subseteq A$ ，in the same way that it is correct to write $5 \leq 5$ ．But should the empty set be counted as a subset of $A$ ？The definition requires that every element of $\emptyset$ be contained in $A$ ，but there are no elements of $\emptyset$ to which we may apply this condition． Technically，we say that the condition is vacuously satisfied．The Mathematical Outing above might provide a more compelling reason to declare that $\emptyset \subseteq A$ ．

At times we may wish to exclude the option of taking the empty set as a subset；in this case we use language like＂Let $A$ be a nonempty subset of $B$ ．＂ On the other hand，to rule out the option of selecting all of $B$ as a subset， we would say＂Let $A$ be a proper subset of $B$ ．＂We indicate this by writing $A \subset B$ ，in the same way that we use $<$ rather than $\leq$ when the two objects being compared are not permitted to be equal．

[^0]As an illustration, let $A=\{2,4,6,8, \ldots\}$ be the set of even natural numbers, let $B=\{1,2,4,8, \ldots\}$ list the powers of 2 , and let $C=\{6,12,18,24, \ldots\}$ contain the multiples of six. Comparing the elements within the various sets, we quickly realize that every multiple of six is even but not vice-versa, hence $C \subset A$. In addition, no multiple of 6 is a power of 2 , hence $B$ and $C$ are disjoint; that is, $B \cap C=\emptyset$. There is a single power of 2 that is odd, which is enough to prevent $B$ from being a subset of $A$, a fact which may be conveyed succinctly as $B \nsubseteq A$. Deleting all the even numbers from set $B$ singles out the lone offending odd number, which is 1 .

Removing all elements from a set $B$ that belong to another set $A$ creates a new set: the set difference $B-A$.

Therefore we may write $B-A=\{1\}$ for the sets described above. Alternately, we might consider $A-B=\{6,10,12,14,18, \ldots\}$, the even numbers that are not
 powers of 2 . Note that it is not necessary for one set to be a subset of another to form their set difference. In general, a set difference $B-A$ may be described as "all $B$ that are not $A$." A Venn diagram for the difference $B-A$ is shown at left. From this diagram it becomes clear that we may also define $B-A$ as $\bar{A} \cap B$.
 also list its elements using curly brackets and an ellipsis.
 What is the relationship between $X$ and $A$ ? between $X$ and $B$ ?

The collection of all subsets of $A$ can be assembled into a single larger set.

We define the power set $\mathcal{P}(A)$ of a set $A$ to be the set of all subsets of $A$, including the empty set and the set $A$ itself.

The motivation for this terminology stems from the fact that when $A$ is a finite set, there are exactly $2^{|A|}$ subsets, so the cardinality of $\mathcal{P}(A)$ is a power of 2 . To explain this phenomenon, imagine building a subset of $A$. There are two choices available for the first element of $A$ : either include it in our subset or leave it out. Regardless of our decision, we are now faced with the same two possibilities for the second element - either include it in our subset or leave it out. Continuing this reasoning for each element of $A$, we find that there are $(2)(2) \cdots(2)=2^{|A|}$ ways to build a subset of $A$, as claimed.

The power set of $A$ is a set whose elements are themselves sets, which takes some getting used to. For starters, one has to pay attention not to mix up the symbols $\in$ and $\subseteq$. Thus if $A=\{b, a, l, o, n, e, y\}$ then it would be appropriate
to write $\{n, o, e, l\} \subseteq A$ ，but not $\{n, o, e, l\} \subseteq \mathcal{P}(A)$ ．Since the subsets of $A$ are elements of $\mathcal{P}(A)$ ，we should instead write $\{n, o, e, l\} \in \mathcal{P}(A)$ ．A subset of $\mathcal{P}(A)$ would look like $\{\{n, o, b, l, e\},\{b, a, y\}\}$ ，for instance．
Dan $_{\text {距 }}^{\text {ETT }}$ h）For the set $A=\{b, a, l, o, n, e, y\}$ ，write down a subset of $\mathcal{P}(A)$ having three elements that are pairwise disjoint，but whose union is all of $A$ ．
Q】 IaRY i）Let $A=\{c, d, e\}$ and let $B=\{a, b, c, d\}$ ．Determine the sets contained in $\mathcal{P}(B-A)$ and the sets contained in $\mathcal{P}(B)-\mathcal{P}(A)$ ．

Suppose that $A$ and $B$ are nonempty sets．As a slightly intricate but very instructive example，let us demonstrate that every element of the power set $\mathcal{P}(B-A)$ is contained in $\mathcal{P}(B)-\mathcal{P}(A)$ ，with one exception．In other words，we will show that $\mathcal{P}(B-A)$ is almost a subset of $\mathcal{P}(B)-\mathcal{P}(A)$ ．The exception is the empty set，for $\emptyset \in \mathcal{P}(B-A)$ but $\emptyset \notin \mathcal{P}(B)-\mathcal{P}(A)$ ．It is true that $\emptyset \in \mathcal{P}(B)$ ， but $\emptyset \in \mathcal{P}(A)$ as well，so it is removed when we subtract $\mathcal{P}(A)$ ．

Now let $X$ be a non－empty subset of $B-A$ ，so that $X \in \mathcal{P}(B-A)$ ．（We write $X$ instead of $x$ ，since we are referring to a set instead of an element．）We will show that $X \in \mathcal{P}(B)-\mathcal{P}(A)$ as well．Since $X \subseteq B-A$ ，each element of $X$ is a member of $B$ but not of $A$ ．In other words，$X$ is a subset of $B$ ，but $X$ and $A$ are disjoint．It follows that $X \in \mathcal{P}(B)$ since $X \subseteq B$ ．But clearly $X \notin \mathcal{P}(A)$ since the elements of $X$ are not in $A$ ．（Here is where we use the fact that $X$ is non－empty．）So when we subtract $\mathcal{P}(A)$ from $\mathcal{P}(B)$ ，the element $X$ of $\mathcal{P}(B)$ is not removed．Hence $X \in \mathcal{P}(B)-\mathcal{P}(A)$ ，as claimed．${ }^{\dagger}$

a）The sets $\{m, a, i, n, e\}$ and $\{t, e, x, a, s\}$ ，for instance．
b）We have $A$ and $B$ disjoint，$A \subseteq B$ ，and $B \subseteq A$ ，respectively．
c）Assuming that the circle representing $B$ is on the left，draw a smaller circle for $A$ within the left circle but outside the right circle．
d）Of course you can！Just extend your hand so that one finger touches your tongue and another finger touches your nose．The point of this seemingly irrelevant exercise is to highlight the fact that we often use and interpret the word AND in a careless manner．If you tried to curl your tongue upwards to accomplish this activity，you were trying to touch your tongue to your nose．

In the same way，it is easy to slip up by attempting to prove that $C \subseteq \bar{A}$ by writing ＂Suppose that $x \in C$ and $x \in \bar{A}$ ．＂But a conjunction is logically quite different from an implication，dooming this proof from the outset．So be careful to phrase a set inclusion proof as an implication－＂We wish to show that if $x \in C$ then $x \in \bar{A}$ ．＂
e）Six subsets of $\{1,2,3\}$ are both nonempty and proper．
f）The set $A-C$ consists of positive even numbers that are not multiples of 6 ，namely the numbers $\{2,4,8,10,14, \ldots\}$ ．
g）This means that $X$ is a subset of $B$ and that $X$ and $A$ are disjoint．
h）One answer could be $\{\{b, o, y\},\{a, l, e\},\{n\}\}$ ．
i）The sets $\{a, b\},\{a\},\{b\}, \emptyset$ make up $\mathcal{P}(B-A)$ ．On the other hand， $\mathcal{P}(B)-\mathcal{P}(A)$ consists of all sixteen subsets of $B$ except for $\{c, d\},\{c\},\{d\}, \emptyset$ ．
益 If we include the empty set，the subsets of $C_{3}$ are $\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\}$ ， $\{2,3\}$ and $\{1,2,3\}$ ．In this fashion we find that the number of subsets of $C_{1}$ to $C_{4}$ are $2,4,8$ and 16 when we include the empty set．These powers of 2 are much more appealing than the alternative，which suggests that $\emptyset \subseteq C_{2}$ should be true．Including the empty set，there are $2^{n}$ subsets of $C_{n}$ ．

## Exercises

32. Let $A=\{f, l, a, t\}$. To remove the letter $l$, do we write $A-l$ or $A-\{l\}$ ?
33. Suppose that $A \subseteq B$. What is the relationship between sets $\bar{A}$ and $\bar{B}$ ?
34. If $A$ is a proper subset of $B$ then what can be said of the set $B-A$ ?
35. Would it be correct to assert that $\emptyset \subseteq \mathcal{P}(A)$ ? Does it make sense to write $\emptyset \in \mathcal{P}(A)$ ? What is the difference between these two statements?
36. Let $A=\{x \mid-1<x<1\}, B=\{x \mid-2 \leq x \leq 2\}$ and $C=\{x \mid-2<x<3\}$, where $x \in \mathbb{R}$. Determine whether the following statements are true or false.
a) $A \subseteq B$ and $B \subseteq C$
b) $\bar{C} \subseteq \bar{B}$ or $\bar{B} \subseteq \bar{A}$
c) $A-C$ is the empty set
d) $C-B=\{x \mid x=-2$ or $2<x<3\}$
e) $\bar{A}$ and $B$ are disjoint
37. Construct two finite sets $A$ and $B$ such that $|B|=7,|A|=5$ and $|B-A|=4$. (Your example shows that in general $|B-A| \neq|B|-|A|$.)
38. Suppose that sets $A$ and $B$ satisfy $|A|=101,|B|=88$ and $|B-A|=31$. Determine $|A-B|$. (Hint: Use a Venn diagram.)
39. Let $A=\{g, n, a, r, l, y\}$. What is the only set that is both a subset of $A$ and disjoint from $A$ ?
40. Let $B=\{b, r, i, d, g, e\}$. How many nonempty subsets of $B$ are disjoint from the set $\{s, t, r, e, a, m\}$ ?
41. If $C=\{s, a, t, i, n\}$, then how sets $D \in \mathcal{P}(\mathcal{C})$ satisfy $|D|=2$ ?
42. Suppose that $A=\{b, i, s, m, a, r, c\}$. How many subsets of $A$ contain $m$ ?
43. Given sets $B=\{t, u, r, k, e, y\}$ and $A=\{b, r, u, t, e\}$, compute $|\mathcal{P}(B)-\mathcal{P}(A)|$.

## Writing

44. For sets $A$ and $B$, show that $A \cap B$ and $B-A$ are disjoint. Give a written proof that does not rely on a Venn diagram.
45. Given sets $A, B$ and $C$, explain why $\overline{B \cup C}$ and $(A \cap B) \cup C$ are disjoint. Do not rely on a Venn diagram in your proof.
46. Prove that $(A \cup B) \cap C \subseteq A \cup(B \cap C)$. Give a written proof that does not rely on a Venn diagram, and also illustrate this result with a Venn diagram.
47. Prove that $(A-B)-C \subseteq A-(B-C)$. Give a written proof that does not rely on a Venn diagram, and also illustrate this result with a Venn diagram.
48. Demonstrate that if $B \subseteq C$ then $A \cup \bar{C} \subseteq A \cup \bar{B}$.
49. Suppose that $A, B$ and $C$ are sets such that $A-B \subseteq C$. Show that in this case $\bar{C} \subseteq \bar{A} \cup B$. Do not rely on a Venn diagram in your proof.
50. For sets $A$ and $B$, explain why $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
51. For sets $A$ and $B$, prove that $\mathcal{P}(A) \cap \mathcal{P}(B)=\mathcal{P}(A \cap B)$.
52. Establish that $\mathcal{P}(B)-\mathcal{P}(A)=\mathcal{P}(B)-\mathcal{P}(A \cap B)$ for any sets $A$ and $B$.
53. Let $C$ be a set of Halloween candies. Suppose that Aaron helps himself to some (possibly none) of the candies, and then Betty does the same with what remains. (There may well be candies left over at the end of this process.) Prove that there are $3^{|C|}$ ways for the candy distribution to take place.

### 2.4 Cartesian Products

It is not at all unusual for a single object, mathematical or otherwise, to have two or more numbers associated with it. For instance, at each visit a pediatrician will record both a child's height in inches and his weight in pounds. This information can be succinctly presented as an ordered pair of numbers, as in $(42,57)$ for a solid seven-year old. We can think of the 42 as an element from the set of all possible heights, and the 57 as an element from the set of all possible weights.

Given two sets $A$ and $B$, their Cartesian product $A \times B$ is the set consisting of all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. When $A$ and $B$ are both finite sets, we have $|A \times B|=|A| \cdot|B|$.

Perhaps the most familiar example of a Cartesian product is the set of points in the Cartesian plane. Such a point has an $x$-coordinate and a $y$-coordinate, which are presented as the ordered pair $(x, y)$. Each coordinate is a real number, so the Cartesian plane is the product $\mathbb{R} \times \mathbb{R}$, sometimes written as $\mathbb{R}^{2}$ for short.

Anyone who has played a game of Battleship has dealt with a Cartesian product. The square game board is divided into a grid, with rows labelled 'A' through ' $J$ ' and columns numbered 1 to 10 . Each location on the board is referred to by a letter and a number, as in "Is (C,7) a hit?" From a mathematical perspective, the locations on the game board represent the Cartesian product of the
 sets $A=\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{I}, \mathrm{J}\}$ and $B=\{1,2,3,4,5,6,7,8,9,10\}$. The elements of this Cartesian product are the pairs

$$
(\mathrm{A}, 1) \quad(\mathrm{A}, 2) \quad(\mathrm{A}, 3) \quad \cdots \quad(\mathrm{J}, 9) \quad(\mathrm{J}, 10) .
$$

These ordered pairs are arranged in a $10 \times 10$ grid, so there are 100 of them, which agrees with the fact that $|A \times B|=|A| \cdot|B|=10 \cdot 10=100$.
 way to list all the elements of $A \times B$. Based on your list, why does it make sense that $|A \times B|=|A| \cdot|B|$ ?

## Mathematical Outing $\star \star \star$

The game of PairMission is played using the ordered pairs of a Cartesian product. To begin, two players select disjoint finite sets $A$ and $B$, such as $A=\{a, b, c, d\}$ and $B=\{1,2,3\}$. The players
 alternate turns writing down ordered pairs in a column, with the rule that a play is "pairmissible" as long as the letter and number in the pair have not both been used earlier in the game. The winner is the last person able to write down a legal ordered pair. For example, the sequence of moves

$$
(a, 2)(b, 2)(a, 3)(c, 2)
$$

is possible. The first player could now win the game by writing down $(d, 1)$, since there are no further legal moves.

Play a few rounds of PairMission to get a feel for the game. How can one quickly ascertain whether the game is over? Now explain how the first player can guarantee a win when $A=\{a, b, c\}$ and $B=\{1,2,3\}$. Then demonstrate that the second player can force a win for $A=\{a, b, c, d\}$ and $B=\{1,2,3\}$. After analyzing a few more games, make a conjecture concerning which player has a winning strategy for any pair of sets $A$ and $B$.

To reinforce these ideas, suppose now that $A=\{x \in \mathbb{R} \mid 3 \leq x \leq 6\}$ and $B=\{y \in \mathbb{R} \mid 1 \leq y \leq 8\}$. Then $A \times B$ would consist of all ordered pairs $(x, y)$ of real numbers for which $3 \leq x \leq 6$ and $1 \leq y \leq 8$. The most natural way to visualize the collection of all such ordered pairs is as a subset of the Cartesian plane. The points ( $x, y$ ) in $A \times B$ constitute a solid rectangle with width 3 and height 7 , pictured in the diagram.

## 

b) Use a diagram to help illustrate why the assertion $(A \times C) \cup(B \times D)=(A \cup B) \times(C \cup D)$ is false. How could one modify the left-hand side to create a valid set identity?


To indicate the sorts of steps needed to prove a statement about Cartesian products, we will show that $(A \times C) \cup(B \times D) \subseteq(A \cup B) \times(C \cup D)$. We must show that every element of the first set is also a member of the second one. But now the sets are Cartesian products, so we represent a generic element as $(x, y)$ rather than just $x$. Thus we suppose that $(x, y) \in(A \times C) \cup(B \times D)$. This means that either $(x, y) \in(A \times C)$ or $(x, y) \in(B \times D)$, so we must consider two separate cases. On the one hand, if $(x, y) \in(A \times C)$ then by definition $x \in A$ and $y \in C$. But since $x \in A$ then clearly $x \in A \cup B$, and similarly $y \in C$ implies that $y \in C \cup D$. Therefore $(x, y) \in(A \cup B) \times(C \cup D)$. The case where $(x, y) \in(B \times D)$ is entirely analogous, so we are done. ${ }^{\dagger}$

Qப
c) Suppose that $A=\{1,2\}$ and $B=\{1,2,3\}$. Write out the elements of $A \times B$ and $B \times A$. Do we obtain the same ordered pairs in each case? In other words, is $A \times B=B \times A$ ?

Since the order in which elements are listed in an ordered pair matters, in general it is not the case that $A \times B$ and $B \times A$ are the same set. However, in a few special cases these two Cartesian products do consist of exactly the same set of ordered pairs. The first possibility is that $B=\emptyset$, for in this case both $A \times B$ and $B \times A$ are the empty set.

## Qப

By the same reasoning, both Cartesian products are empty when $A=\emptyset$ as well.
If both $A$ and $B$ are nonempty, there is only one other way to ensure that $A \times B=B \times A$. To discover what this condition might be, let's take any elements $x \in A$ and $y \in B$. (This is possible since neither $A$ nor $B$ are the empty set.) So we would have $(x, y) \in A \times B$. But since $A \times B=B \times A$, we could then write $(x, y) \in B \times A$, which means that $x \in B$ and $y \in A$. In summary, we deduce that if $x$ is any element of $A$ then $x \in B$ also, and furthermore that if $y$ is any element of $B$ then $y \in A$. But this is exactly our criteria for showing that two sets are equal, so we conclude that we must have $A=B$. Clearly this condition works, for both sides of $A \times B=B \times A$ reduce to just $A \times A$.

By the way, taking the Cartesian product of a set with itself is a fairly common occurrence in mathematics; we have already seen one example above when we wrote the plane as $\mathbb{R} \times \mathbb{R}$. The set $A \times A$ will also play an important role when we discuss relations and functions in a later chapter.

a) List all the ordered pairs in a three by four table. The top row would contain $(10,1)(10,2)(10,3)(10,4)$, and so on.
b) Replace the left-hand side by including two extra Cartesian products in the union: $(A \times C) \cup(A \times D) \cup(B \times C) \cup(B \times D)$.
c) The ordered pairs are not identical; for instance, $(1,3) \in A \times B$ but $(1,3) \notin B \times A$. Thus $A \times B \neq B \times A$ for these sets.
d) If there are no elements in $B$, then there is no way to create an ordered pair $(x, y)$ with $y \in B$.
Ill The game is over as soon as all available letters and numbers appear at least once somewhere in the list of moves. When $A=\{a, b, c\}$ and $B=\{1,2,3\}$, suppose the first player writes down $(a, 1)$. If the second player matches neither of these characters, say by playing $(b, 3)$, then the first player should take the remaining two characters, which are $(c, 2)$ to win the game. However, if the second player does match one of the characters, say by playing $(b, 1)$, then the first player should continue to match that character by playing $(c, 1)$. The game must now last for exactly two more moves, causing the first player to win in this scenario as well.

Analysis of the game with $A=\{a, b, c, d\}$ and $B=\{1,2,3\}$ is left to the reader. In general, it turns out that the second player has a winning strategy if at least one of $|A|$ and $|B|$ is even, while the first player can always win if both $|A|$ and $|B|$ are odd. (Can you figure out the winning strategies?) Finally, try to find a nice way to represent this game by putting markers on a rectangular grid whose rows are labeled with the letters in $A$ and whose columns are labeled by the numbers in $B$.

## Exercises

54. What can we deduce about sets $A$ and $B$ if $A \times B=\emptyset$ ?
55. Write the definition of the Cartesian product $A \times B$ using bar notation.
56. Explain how a standard deck of cards illustrates a Cartesian product.
57. Let $U=\{1,2, \ldots, 9\}$ be the universal set, and let $A=\{n \mid n \in U, n$ is odd $\}$ and $B=\{n \mid n \in U, n$ is a perfect square $\}$. Compute the cardinality of
a) $U \times U$
b) $A \times B$
c) $\bar{A} \times \bar{B}$
d) $\overline{A \times B}$
e) $(A \cup B) \times(A \cap B)$
f) $(A-B) \times(\bar{B}-\bar{A})$
58. Do you believe that $\overline{A \times B}=\bar{A} \times \bar{B}$ based on your answers to the previous exercise? Why or why not?
59. Let $S$ be the subset $\left\{(x, y) \mid x^{2}+2 y^{2}=10\right\}$ of $\mathbb{R}^{2}$. What is the common name for this mathematically defined set?
60. Suppose $C=\{w \mid 1 \leq w \leq 3\}$ and $D=\{w \mid 2 \leq w \leq 5\}$. Then $C \times D$ is a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.
a) Sketch the region corresponding to $C \times D$ and describe its shape.
b) Draw the subset $D \times C$ on the same set of axes.
c) Use your diagram to determine $(C \times D) \cap(D \times C)$.
61. Define the sets $A=\{s, c, a, m, p, e, r\}, B=\{p, r, a, n, c, e\}, C=\{1,2,3,5,8\}$ and $D=\{2,3,5,7,11,13\}$. Describe the intersection $(A \times C) \cap(B \times D)$.
62. Craft a verbal description of the Cartesian product $A \times B \times C$.
63. Suppose that $A=\{1,2,3,4,5\}, B=\{a, b, c, d, e, f\}$ and $C=\{\bullet, \star, \diamond\}$. How many elements are there in the set $A \times B \times C$ ?

## Writing

64. Show that for two sets $A$ and $B$ within some universal set $U$, it is the case that $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$.
65. Prove that $A \times(B \cup C)=(A \times B) \cup(A \times C)$ for any three sets $A, B$ and $C$.
66. Make and prove a conjecture regarding the relationship between the sets $A \times(B-C)$ and $(A \times B)-(A \times C)$.
67. Demonstrate that $(A \times C) \cap(B \times D)=(A \times D) \cap(B \times C)$ for any four sets $A, B, C$ and $D$.
68. The intersection $(A \times B) \cap(B \times A)$ can be written as the Cartesian product of a certain set with itself. Find, with proof, an expression for that set.
69. Let $A=\{1,2,3, \ldots, 10\}$. Prove that if we select any twenty ordered pairs from $A \times A$, then we can always find two of the chosen pairs that give the same sum when the numbers within the pair are added together.
70. Prove that the game of PairMission described in the Mathematical Outing for this section will end after at most $|A|+|B|-1$ moves.

### 2.5 Index Sets

For sake of illustration, consider the set of all words that contain the letter ' $a$ '. For our purposes a word may be formed from any finite string of lower case letters from our alphabet, such as 'gargantuan' or 'scrambleflopsy' or 'sjivkavl.' Naturally we would also be interested in the set of all words that contain the letter b, or the letter $c$, and so on. In this sort of situation it makes sense to name each set in a manner that reflects the letter on which it depends.

To accomplish this task we employ subscripts. Thus we let $W_{\mathrm{a}}$ be the set of all words containing the letter a, and similarly for $W_{\mathrm{b}}$ through $W_{\mathrm{z}}$. The common variable name $W$ reflects the fact that each set contains words.

The subscripts $\mathrm{a}, \mathrm{b}, \ldots$ are known as indices; the set $I=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots, \mathrm{z}\}$ of all indices is called the index set. The collection of all the sets $W_{\mathrm{a}}$ through $W_{z}$ comprises a family of sets, in the sense that they are related by a common definition. It may help to remember that each index indicates a particular set in the family.
 divisible by 5 . Of course, there is a whole family of such sets. Decide on a name for these sets and identify the index set.

Observe that we are not introducing any new set operations in this section. Rather, we are describing a scheme for organizing related sets. But we are free to apply set operations to indexed sets-they are just sets, after all. Thus $\bar{W}_{\mathrm{p}}$ is the set of words which do not contain the letter p , so 'rambunctious' $\in \bar{W}_{\mathrm{p}}$, for instance. Furthermore, 'chocolate' $\in W_{\mathrm{c}} \cap W_{\mathrm{t}}$ and 'zamboni' $\in W_{\mathrm{z}}-W_{\mathrm{e}}$.

It may come as a surprise to learn that the intersection of all the sets $W_{\alpha}$ for $\alpha \in I$ is non-empty. As you might expect, the intersection of an indexed collection of sets consists of those elements that appear in every single set. We could write their intersection as $W_{\mathrm{a}} \cap W_{\mathrm{b}} \cap \cdots \cap W_{\mathrm{z}}$, but this notation is cumbersome at best. Instead we adopt the notation $\bigcap_{\alpha \in I} W_{\alpha}$. Hence our definition of the intersection of the family of sets $W_{\alpha}$ can be shortened to

$$
\bigcap_{\alpha \in I} W_{\alpha}=\left\{x \mid x \in W_{\alpha} \text { for all } \alpha \in I\right\} .
$$

If this intersection is to be non-empty, then there must exist a word that contains every letter of the alphabet at least once! This does seem surprising, until we remember that in the present setting 'words' are arbitrary strings of letters, not necessarily English words. For instance, we have

$$
\text { 'aquickfoxjumpsoverthelazybrowndog' } \in \bigcap_{\alpha \in I} W_{\alpha} \text {. }
$$


b) What is $\bigcap_{\alpha \in I} \bar{W}_{\alpha}$ ?

In order to have other examples of indexed sets at our disposal，we now formally define two frequently encountered sets of real numbers．

An open interval is a set of real numbers of the form $\{x \in \mathbb{R} \mid a<x<b\}$ ， for fixed real numbers $a<b$ ，while a closed interval is a set of the form $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ ．These sets are usually written compactly as $(a, b)$ and $[a, b]$ ，respectively．

Although one might worry that the ordered pair $(3,7)$ could be confused with the open interval $(3,7)$ ，in practice this hardly ever occurs．Several examples of open and closed intervals are pictured below．It is often useful to have a mental picture such as this in mind when working with intervals．

 secting two closed intervals？What are the possibilities when working with open intervals instead？

Having introduced the notion of an interval，we are now prepared to consider an entire family of open intervals．Let $B_{r}$ be the set of all numbers between -1 and some positive real number $r$ ．In other words，define $B_{r}=(-1, r)$ ．The first two intervals pictured above are members of this family；namely，$B_{3}$ and $B_{4.5}$ ． In this case the index set is $J=\{r \mid r \in \mathbb{R}, r>0\}$ ．It is important to make a distinction between the index set and the other sets belonging to the family． Think of the index set $J$ as the master set：it catalogs all the other sets，since there is one set $B_{r}$ for each $r \in J$ ．
（品唁䍚T d）Based on the definition of $B_{r}$ ，which of the following are correct？ $\begin{array}{lll}\text { i．} B_{3} \subset B_{7} & \text { ii．} \sqrt{10} \in B_{\pi} & \text { iii．} B_{4.5} \text { and } \bar{B}_{5.2} \text { are disjoint．}\end{array}$

In the same manner as before we can form the intersection $\bigcap_{r \in J} B_{r}$ of the entire family of open intervals．The challenge in this case is not so much under－ standing the notation as determining the answer．Clearly $-\frac{1}{2}$ is contained in every such interval，as is -0.1 ．In fact，every real number from -1 up to and including 0 is contained in the interval $B_{r}$ for all $r \in J$ ．（Convince yourself of this fact．）But no positive real number is contained in every set $B_{r}$ for all $r>0$ ． For example，consider the number .0001 ．We need only select a smaller positive number，such as $r=.000001$ ，in order to find a set that does not contain .0001 ．

## Mathematical Outing $\star \star \star$

For each $n \in \mathbb{N}$, let $C_{n}$ be the set of counting numbers from 1 to $n$, so that

$$
C_{n}=\{1,2, \ldots, n\} .
$$



To begin, show that $\bigcup_{n=1}^{\infty} C_{n}=\mathbb{N}$. (This is a set equality, so explain why every element of $\bigcup_{n=1}^{\infty} C_{n}$ is in $\mathbb{N}$ and vice-versa.) What do subsets of $C_{n}$ look like? In other words, describe the elements of $\mathcal{P}\left(C_{n}\right)$. Next describe the set $\bigcup_{n=1}^{\infty} \mathcal{P}\left(C_{n}\right)$ in a single complete sentence. Now for the stumper: is this set the same as $\mathcal{P}(\mathbb{N})$ ? Why or why not?

And since $.0001 \notin B_{.000001}$, it is not contained in the intersection of all the $B_{r}$. We conclude that

$$
\bigcap_{r \in J} B_{r}=\{x \mid-1<x \leq 0\} .^{\dagger}
$$

Just as we can find the intersection of a family of sets, we can also find their union. For example, consider the union of the sets $W_{\alpha}$, where $\alpha$ represents a letter of the alphabet. Predictably, such a union is written in the form $\bigcup_{\alpha \in I} W_{\alpha}$, and consists of those words that are members of at least one of the sets $W_{\alpha}$. But every word is a member of some $W_{\alpha}$, since every word contains at least one letter, so the union is the set of all words.

It is quite common for a family of sets to be indexed by simply being numbered. Whenever our index set is the natural numbers (or a subset thereof) there is a more informative way of writing an intersection or union, reminiscent of sigma notation. For instance, imagine that we had a family $A_{1}, A_{2}, A_{3}, \ldots$ of sets. We can express the intersection of sets $A_{3}$ through $A_{6}$ as

$$
\bigcap_{n=3}^{6} A_{n}=A_{3} \cap A_{4} \cap A_{5} \cap A_{6} .
$$

Similarly, the union of sets $A_{1}$ through $A_{7}$ is written $\bigcup_{n=1}^{7} A_{n}$, so that

$$
\bigcup_{n=1}^{7} A_{n}=A_{1} \cup A_{2} \cup A_{3} \cup A_{4} \cup A_{5} \cup A_{6} \cup A_{7} .
$$

If we wish to take the intersection of all the sets in the family we employ the notation $\bigcap_{n=1}^{\infty} A_{n}$, just as is done for sigma notation when expressing an infinite series. Predictably, an infinite union is written as $\bigcup_{n=1}^{\infty} A_{n}$. This notation is quite versatile; thus the union of all the even-numbered sets can be written $\bigcup_{n=1}^{\infty} A_{2 n}$, while the intersection of all odd-numbered sets is $\bigcap_{n=1}^{\infty} A_{2 n-1}$.

ML THE $\begin{aligned} & \text { THSWERS }\end{aligned}$
a) Let $M_{n}$ be the set of all integers that are multiples of $n$. The index set is $\mathbb{N}$, since we have one set $M_{n}$ for each $n \in \mathbb{N}$.
b) The empty set, since no word omits every letter of the alphabet. c) The intersection of two closed intervals is either a closed interval, a point, or the empty set. Two open intervals intersect in either an open interval or the empty set.
d) $i$. True $i$. False, since $\sqrt{10}>\pi$ iii. True
e) Every real number $x>-1$, no matter how large, is a member of some set $B_{r}$. For example, 1000000 is an element of $B_{1000001}$. Hence $\bigcup_{r \in J} B_{r}=\{x \mid x>-1\}$.
興 Argue that if $x \in \bigcup_{n=1}^{\infty} C_{n}$ then $x \in C_{k}$ for some $k$, which means that $x$ is a counting number from 1 to $k$, so $x \in \mathbb{N}$. Conversely, if $x \in \mathbb{N}$ then $x$ is a counting number, say $x=k$. But then $x \in C_{k}$ (and $C_{k+1}$, etc.), and hence $x \in \bigcup_{n=1}^{\infty} C_{n}$.

Elements of $\mathcal{P}\left(C_{n}\right)$ are subsets of $\{1,2, \ldots, n)$; i.e. sets all of whose elements are counting numbers from 1 to $n$. Hence $\bigcup_{n=1}^{\infty} \mathcal{P}\left(C_{n}\right)$ consists of all finite sets of counting numbers. And therein lies the rub, for $\mathcal{P}(\mathbb{N})$ contains all subsets of $\mathbb{N}$, including infinite ones. More concretely, $\{1,3,5,7, \ldots\}$ belongs to $\mathcal{P}(\mathbb{N})$ but not to $\bigcup_{n=1}^{\infty} \mathcal{P}\left(C_{n}\right)$.

## ExERCISES

71. Consider the set of points in the plane a distance $r$ from the origin, where $r$ is a particular real number not smaller than 2. Create an appropriate name for this family of sets and identify the index set.
72. In the previous exercise, use compact notation to indicate the union of all the sets for which $3 \leq r \leq 5$. Also, draw a sketch of this union.
73. Let $A_{k}$ be a family of sets, one set for each element $k \in I$ for some index set $I$. Write a definition for the union of this family using bar notation, as was done for intersection earlier in this section.
74. Let $W_{\alpha}$ be the family defined in this section. Give an example of an element in each of the following sets. (Any string of letters will do, but ordinary English words are cooler.)
a) $W_{\mathrm{x}}-W_{\mathrm{e}}$
b) $W_{\mathrm{j}} \cup W_{\mathrm{q}} \cup W_{\mathrm{v}}$
c) $\overline{W_{\mathrm{a}}} \cap \overline{W_{\mathrm{e}}} \cap \overline{W_{\mathrm{o}}} \cap \overline{W_{\mathrm{u}}}$
d) $\bigcap_{\alpha \in I^{\prime}} W_{\alpha}$, where $I^{\prime}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}\}$
75. Give a description of the elements of $\bigcup_{\alpha \in I^{\prime}} W_{\alpha}$, where $I^{\prime}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}\}$.
76. Write the following sets using sigma-style notation.
a) $A_{8} \cup A_{9} \cup A_{10} \cup A_{11} \cup A_{12}$
b) $B_{3} \cap B_{6} \cap B_{9} \cap \cdots$
c) $C_{2} \cup C_{3} \cup C_{4} \cup \cdots$
d) $D_{5} \cap D_{6} \cap D_{7} \cap D_{8}$
77. Let $D_{n}$ be the set of positive divisors of a natural number $n$. Thus $D_{1}=\{1\}$ and $D_{10}=\{1,2,5,10\}$. Find the following sets, writing them in list form.
a) $\bigcup_{n=14}^{16} D_{n}$
b) $D_{100}-D_{50}$
c) $\bigcup_{n=1}^{100} D_{n}$
d) $\bigcap_{n=1}^{\infty} D_{n}$
78. Let $J=\{r \mid r \geq 1\}$, and for each real number $r \in J$ define $C_{r}$ to be the closed interval $[r, 2 r]$. Sketch the sets $C_{4}, C_{5}$ and $C_{6}$ on a number line. Based on your sketch, find $\bigcap_{k=4}^{6} C_{k}$ and $\bigcup_{k=4}^{6} C_{k}$, writing your answer in bar notation.
79. Using the notation of the previous exercise, determine $\bigcap_{r \in J} C_{r}$ and $\bigcup_{r \in J} C_{r}$.
80. Continuing the previous exercises, let $J^{\prime}=\{x \mid 3 \leq x \leq 4.5\}$. Now determine $\bigcap_{r \in J^{\prime}} C_{r}$ and $\bigcup_{r \in J^{\prime}} C_{r}$.
81. Let $M_{n}$ be the set of integers that are multiples of $n$, where $n$ is a natural number. For instance, $M_{5}=\{\ldots,-10,-5,0,5,10, \ldots\}$ is the set appearing earlier in this section. Determine $M_{3} \cap M_{5}, M_{4} \cap M_{6}$ and $M_{10} \cap M_{15} \cap M_{20}$.
82. Continuing the previous exercise, find a succinct way to describe the intersection $M_{a} \cap M_{b}$, where $a$ and $b$ are positive integers. Also, what is $\bigcap_{n \in \mathbb{N}} M_{n}$ ?
83. Let $B_{r}=(r, 10)$ for $r \leq 8$ be a family of open intervals. Determine $\bigcap_{2<r<4} B_{r}$,
writing your answer in bar notation.
84. For the sets $B_{r}$ in the previous exercise, create an intersection which results in the open interval $(6,10)$.

## Writing

85. Let $A_{t}$ be a family of sets, where $t \in I$. Prove that $\bigcap_{t \in I} A_{t}=\bigcup_{t \in I} \bar{A}_{t}$.
86. For each real number $r$ in the open interval $(0,1)$ let $B_{r}$ be the open interval $(5+r, 8+r)$. Prove that $\bigcap_{r \in(0,1)} B_{r}=[6,8]$.
87. Let $M_{n}$ be the set of integers that are multiples of $n$, where $n$ is a natural number. For instance, $M_{5}=\{\ldots,-10,-5,0,5,10, \ldots\}$. Determine the elements of the set $\mathbb{Z}-\bigcup_{n=1}^{\infty} M_{2 n+1}$ and explain why your answer is correct.

### 2.6 Reference

As before, the purpose of this section is to provide a condensed summary of the most important facts and techniques from this chapter, as a reference when studying or working on material from later chapters. We also include a list of the various strategies we have developed for proving statements about sets.

- Vocabulary set, element, empty set, cardinality, bar notation, intersection, union, universal set, complement, equal sets, set identity, Venn diagram, subset, superset, disjoint, nonempty, proper subset, set difference, power set, Cartesian product, ordered pair, indices, index set, family, open interval, closed interval
- Sets Sets may be described via a verbal description or by listing their elements between curly brackets $\{\cdots\}$. The order in which elements are listed does
not matter, as long as each element is listed only once. One can also employ bar notation, which involves writing down the objects in the set followed by a description of those objects, separated by a bar, as in $\left\{n^{2} \mid n \in \mathbb{N}, n\right.$ odd $\}$ for the squares of the odd numbers. The cardinality of a finite set is the number of elements in the set; hence $|\emptyset|=0$. The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ refer to the natural numbers, integers, rational, real, and complex numbers. Observe that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
- Set Operations We may take the union, intersection, difference or Cartesian product of two sets, denoted by $A \cup B, A \cap B, A-B$ and $A \times B$. We may also take the complement of a set relative to some universal set. Seemingly different combinations of sets may produce the same result, like $\overline{A-B}=\bar{A} \cup B$, giving a set identity. A Venn diagram provides a useful way to visualize combinations of sets and to prove set identities involving two or three sets.
- Power Sets The power set of $A$, written $\mathcal{P}(A)$, is the set of all subsets of $A$, including $\emptyset$ and $A$. When $A$ is finite, there are $2^{|A|}$ subsets of $A$, and therefore $2^{|A|}$ elements of $\mathcal{P}(A)$. If $X$ is a subset of $A$ then we would write $X \subseteq A$ or $X \in \mathcal{P}(A)$, but not $X \subseteq \mathcal{P}(A)$. If the subset $X$ is not equal to all of $A$ then we call $X$ a proper subset, and write $X \subset A$.
- Cartesian Products The Cartesian product $A \times B$ is the set of all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$. When $A$ and $B$ are both finite, there are $|A| \cdot|B|$ such ordered pairs. The Cartesian plane is $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^{2}$ for short. One effective way to visualize a Cartesian product is to organize its elements into a two-dimensional array, with the rows and columns corresponding to the elements of $A$ and $B$. The Cartesian product is not commutative, meaning that in general $A \times B \neq B \times A$.
- Index Sets When a collection of sets that depend on a particular quantity are defined in a common fashion we have a family of sets, such as $A_{k}$ for $k \in I$. The index set $I$ keeps track of all the possible values of the quantity. There is one set $A_{k}$ for each element $k \in I$. We may form the intersection or union of the entire family by writing $\bigcap_{k \in I} A_{k}$ or $\bigcup_{k \in I} A_{k}$. When the sets are numbered, an alternate sigma-style notation may be employed, as in $\bigcup_{k=1}^{5} A_{k}$ or $\bigcap_{k=1}^{\infty} A_{k}$.


## Proof Strategies

The paragraphs below briefly outline strategies for approaching various assertions involving sets, complete with a template for writing a proof. Keep in mind that no such list can possibly be comprehensive; the reader will need to adapt the strategies and templates here to suit the particular statement to be proved.

* Unions and Intersections To show that an element is contained in the union of two sets, it suffices to show that the element is in either set. However, to demonstrate that the element is in their intersection, you must prove that the element is contained in both sets.

On the flip side, if we are given an element in an intersection, we know it is contained in both sets. It is less helpful if we only know that an element is in the union of two sets, since then it may be in either set. In this situation we resort to a proof by cases. If the argument for the two cases is essentially the same, it is acceptable to omit the second case, as illustrated below. (But if the second case is not analogous to the first, be sure to write it out.)

One could write "Since $x \in A \cup B$, we know that either $x \in A$ or $x \in B$. In the first case we have $x \in A$, so [main argument here], which shows that [conclusion]. The case $x \in B$ is analogous, again giving [conclusion]."

* Set Equality To prove that two sets are equal, show that each element in the first set is included in the second set, and vice-versa. These two arguments are usually given in separate paragraphs, unless the proofs are relatively short or very similar in nature.
"We begin by showing that if $x \in$ [first set] then $x \in$ [second set]. [Proof of this statement.] Conversely, it is also true that if $x \in$ [second set] then $x \in$ [first set], because [proof of this statement]. Therefore we may conclude that [first set] $=[$ second set]."

Note that for set identities involving two or three sets, it is also sufficient to create Venn diagrams for each set, justifying in full sentences why you shaded in particular regions, and then observe that the two diagrams are identical. This amounts to a proof by cases, presented visually. This approach applies to any identity involving three or fewer sets, unions, intersections, complements, and set differences. However, don't use Venn diagram proofs for statements involving four or more sets, power sets, or Cartesian products.

* Set Inclusion To prove that one set is contained in another, show that if some object is an element of the first set then it is also an element of the second set. Be aware that it is tempting to begin a proof that $A \subseteq B$ by writing "We must show that $x \in A$ and $x \in B$." This is a logically invalid approach. To demonstrate that $A \subseteq B$ we must prove an implication: if $x \in A$ then $x \in B$.

A sample argument reads as follows "Suppose that $x \in$ [first set]. We must show that $x \in$ [second set]. [Main argument here], which shows that $x \in$ [second set]. It follows that [first set] $\subseteq$ [second set]."

* Set Inequality and Disjoint Sets To prove that two sets are not equal, it suffices to produce a single element that is in one of the sets but not the other.
"To see that these sets are not necessarily equal, consider [give examples of sets]. Then the element $x=$ [counterexample] is in the first set, but not the second, because [reasons]. Therefore the sets are not equal."

To prove that two sets are disjoint, show that given any element of the first set, it cannot also be an element of the second set.
"To prove that the given sets are disjoint, consider any $x \in$ [first set]. Then $x \notin$ [second set] because [reasons]. Therefore the sets are disjoint."
$\star$ TIP $\star$ When deciding what strategy to apply to a particular proof, you must focus on what you are being asked to prove. (This is the statement following the word 'then' in most problems.) Don't get side-tracked by other information at this stage. For instance, suppose that we wish to prove that
Claim: For all sets $A$ and $B$, if $A-B=\emptyset$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
At first glance there are quite a few distracting features to this question: a set equality, a set inclusion, an empty set, power sets, and so forth. With practice, you will learn to immediately concentrate on the expression $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ since it follows the word 'then.' We now recognize that we are being asked to prove a set inclusion, so we can safely write down the first sentence:

Proof: We wish to prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Therefore let $X \in \mathcal{P}(A)$; we wish to show that $X \in \mathcal{P}(B)$.

From our work with power sets we know how to interpret $X \in \mathcal{P}(A)$, so we can also safely write down the second sentence of the proof.

Proof: We wish to prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Therefore let $X \in \mathcal{P}(A)$; we wish to show that $X \in \mathcal{P}(B)$. Equivalently, we have $X \subseteq A$ and we wish to show that $X \subseteq B$.

The proof is now off to a promising start, which is at least a third of the battle. We leave the details of the remainder of the proof to the reader. There is a hint as to how to proceed in the answers at the back.

## SAMPle Proofs

The following proofs provide concise explanations for results discussed within this chapter. They are meant to serve as an illustration for how proofs of similar statements could be phrased. The boldface numbers indicate the section containing each result; the location of that result within the section is marked by a dagger ( $\dagger$ ).
2.2 For sets $A$ and $B$ prove that $\overline{A \cap B}=\bar{A} \cup \bar{B}$.

Proof We begin by showing that each element of $\overline{A \cap B}$ is contained in $\bar{A} \cup \bar{B}$. So suppose that $x \in \overline{A \cap B}$; this means that $x \notin A \cap B$. Since $x$ is not in their intersection, it must lie outside of at least one of $A$ or $B$, hence $x \notin A$ or $x \notin B$. This means that $x \in \bar{A}$ or $x \in \bar{B}$, giving $x \in \bar{A} \cup \bar{B}$.

On the other hand, suppose that we have $x \in \bar{A} \cup \bar{B}$. This means that $x \in \bar{A}$ or $x \in \bar{B}$, so $x \notin A$ or $x \notin B$. Since $x$ is not contained in at least one of $A$ or $B$, it does not reside in their intersection, thus $x \notin A \cap B$. It follows that $x \in \overline{A \cap B}$, as desired. Since the elements of each set are contained in the other, we conclude that $\overline{A \cap B}=\bar{A} \cup \bar{B}$.

2.3 For any sets $A, B$ and $C$, prove that if $A \subseteq B \cap \bar{C}$ then $C \subseteq \bar{A}$.

Proof We wish to prove that $C \subseteq \bar{A}$, so we will show that whenever $x \in C$ then $x \in \bar{A}$ as well. Thus suppose that $x \in C$, which means that $x \notin \bar{C}$. It follows that $x \notin B \cap \bar{C}$ either, since $x$ does not belong to both sets. But if $x \notin B \cap \bar{C}$ while $A \subseteq B \cap \bar{C}$ then we may deduce that $x \notin A$, because $x$ lies outside $B \cap \bar{C}$ while all of $A$ is contained within $B \cap \bar{C}$. Therefore $x \in C$ implies that $x \notin A$, i.e. $x \in \bar{A}$, giving $C \subseteq \bar{A}$.
2.3 Given nonempty sets $A$ and $B$, prove that every set in $\mathcal{P}(B-A)$ is contained in $\mathcal{P}(B)-\mathcal{P}(A)$, except for the empty set.

Proof To begin, we will consider the empty set. We know that $\emptyset \in \mathcal{P}(B-A)$ since the empty set is a subset of every set. However, the empty set is not in $\mathcal{P}(B)-\mathcal{P}(A)$. It is true that $\emptyset \in \mathcal{P}(B)$, but $\emptyset \in \mathcal{P}(A)$ as well, so it is removed when we subtract $\mathcal{P}(A)$ from $\mathcal{P}(B)$.

Now suppose that $X$ is any nonempty set in $\mathcal{P}(B-A)$. We wish to argue that $X \in \mathcal{P}(B)-\mathcal{P}(A)$, which means we must prove that $X \in \mathcal{P}(B)$ but $X \notin \mathcal{P}(A)$. By definition of power set, this is the same as showing that if $X \subseteq B-A$ then $X \subseteq B$ but $X \nsubseteq A$. So suppose that $X \subseteq B-A$. This means that every element of $X$ is in $B-A$, i.e. is in $B$ but not in $A$. Since every element of $X$ is in $B$ we do have $X \subseteq B$. However, since all elements of $X$ are not in $A$ (of which there is at least one, since $X$ is nonempty), we also have $X \nsubseteq A$. Hence if $X \in \mathcal{P}(B-A)$ it follows that $X \in \mathcal{P}(B)-\mathcal{P}(A)$, as claimed.
2.4 Prove that $(A \times C) \cup(B \times D) \subseteq(A \cup B) \times(C \cup D)$ for sets $A, B, C$ and $D$.

Proof To prove this set inclusion we show that if $(x, y) \in(A \times C) \cup(B \times D)$ then $(x, y) \in(A \cup B) \times(C \cup D)$. So suppose that $(x, y) \in(A \times C) \cup(B \times D)$. Since $(x, y)$ is an element of a union of sets, we know that either $(x, y) \in A \times C$ or $(x, y) \in B \times D$. We consider each possibility separately. In the first case we have $(x, y) \in A \times C$, hence $x \in A$ and $y \in C$. But this implies that $x \in A \cup B$ and $y \in C \cup D$ by definition of union. It follows that $(x, y) \in(A \cup B) \times(C \cup D)$, as desired. The proof of the second case in which $(x, y) \in B \times D$ is analogous, so we are done.

2.5 Let $B_{r}$ represent the open interval $(-1, r)$ and let $J$ be the set of positive real numbers. Describe, with proof, the set $\bigcap_{r \in J} B_{r}$.
Proof We claim that the intersection of this family consists of all $-1<x \leq 0$. First, if $x \leq-1$ then $x \notin B_{r}$ for any $r$ according to the definition of $B_{r}$, and thus $x$ is clearly not in their intersection. Furthermore, if $-1<x \leq 0$ then $x \in B_{r}$ for every positive real number $r$, since $B_{r}$ consists of all real numbers between -1 and $r$, which certainly includes any $x$ in the range $-1<x \leq 0$. Hence these values of $x$ belong to the intersection $\bigcap_{r \in J} B_{r}$. Finally, given any $x>0$, choose $r=\frac{1}{2} x$. Then $r$ is a smaller positive real number, so $x \notin B_{r}$ for this particular $r$. Since $x$ is absent from at least one such set, it does not belong to their intersection. In summary, $\bigcap_{r \in J} B_{r}=\{x \in \mathbb{R} \mid-1<x \leq 0\}$.


[^0]:    

