Level Three Arithmetic Graph Sums

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Abstract

A level three graph sum $S(G)$ is the sum over all three-colorings of the vertices of a given multigraph G of the "total edge value" for each coloring. Each edge is assigned a cube root of unity depending upon the colors of its endpoints; the product over all edges gives the corresponding term in the sum. This assignment may be defined via a symmetric 3×3 matrix whose entries are cube roots of unity. We demonstrate that the 729 possible matrices lead to 31 nonequivalent level three graph sums, then pinpoint 81 of them (representing 7 equivalence classes) with the nice property that $S(G)$ gives a power of three for every graph G; these are the arithmetic graph sums. We propose three characterizations of matrices that give arithmetic graph sums, then prove that these seven graph sums have the desired property.

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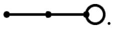
1 Introduction

To illustrate the sorts of results obtained in this paper we begin by presenting the following example. Let G be the graph consisting of a path with two edges having a loop at one end, so that G looks like . Next label each of the vertices with a number from the set $\{0, 1, 2\}$ and assign the cube root of unity $\omega^{q_1 q_2}$ to each edge, where q_1 and q_2 are the numbers labeling the endpoints of the edge and $\omega = e^{2\pi i/3}$. Thus numbering the vertices 0, 1 and 2 from left to right yield edges with values ω^0, ω^2 and ω^4, again from left to right. The product of all the edge values reduces to 1 in this case, the "total edge value" for this labeling. Adding the total edge values over all 27 possible labelings of the vertices gives $3 + 6\omega$. This sum is more than just a multiple of 3; taking the norm gives $\mathcal{N}(3 + 6\omega) = 27$, a power of 3. It is a remarkable (and rather non-trivial) fact that this type of sum always yields powers of 3 for any multigraph G.

Our purpose is to enumerate and characterize all graph sums with this arithmetic property. However, looking beyond the scope of the present paper momentarily, the ideas we begin to develop here should have important applications to graph theory. Each arithmetic graph sum produces a value for a given graph G; if the set of all such values is properly organized then it is not inconceivable that the resulting mathematical object could provide a complete invariant for multigraphs or otherwise encode information about G. This has been an ongoing theme in graph theory: see [3], [4], or [5], among many others.

At the very least, the results are so compelling that they must be pointing towards something greater beyond. For now I will be content with a moderately complete understanding of one aspect of this subject, namely the level three graph sums. I follow Diestel [2] for basic notation and terminology, so $|G|$ denotes the number of vertices of a graph G, and 'induced subgraph' refers to a vertex subgraph. For the most part I will not consider empty graphs, although subgraphs may be empty. Finally, all congruences are mod 3, although the modulus is omitted at times.

2 Motivation

The notion of a graph sum is a natural extension of the following observation. Let G be a simple graph with loops permitted; that is, a finite collection of vertices along with a set of edges which join distinct pairs of these vertices, or which join a vertex to itself. An induced subgraph $G' \subset G$ consists of a subset of the vertices of G, possibly empty, along with all edges of G both of whose endpoints are contained in this subset. Then the number of such subgraphs of G with an even number of edges and the number of subgraphs with an odd number of edges are either equal or differ by a power of 2. More precisely, we have

Proposition 1 *With terminology as above, let $e(G')$ denote the number of edges of G' and define*

$$S(G) = \sum_{G' \subset G} (-1)^{e(G')}. \tag{1}$$

Then either $S(G) = 0$ or $S(G) = \pm 2^m$ for some positive integer m satisfying $\left\lfloor \frac{|G|+1}{2} \right\rfloor \leq m \leq |G|$, where $|G|$ denotes the number of vertices of G.

This result is illustrated in Fig. 1 for a graph G on four vertices having six edges, one of which is a loop. Evidently $S(G) = 12 - 4 = 8$ in this case. Like many results in graph theory, this deceptively simple statement is more resistant to proof than one might expect. The reader is invited to supply a short, clean proof. We outline one possible approach later on in the appendix.

The route to generalizing these ideas lies in the realization that each term in (1) may be obtained by two-coloring the vertices of G, say black and white, and then assigning a value to each edge of G depending upon the color of its endpoints. In this case edges joining black vertices are given a value of -1, while all others are given a value of $+1$. The graph sum then arises as the sum over all vertex colorings of the product of the values of all the edges. This process is illustrated in Fig. 1, in which grey edges have value $+1$ and black edges have value -1. Finally, we observe that the edge values of $+1$ and -1 are square roots of unity.

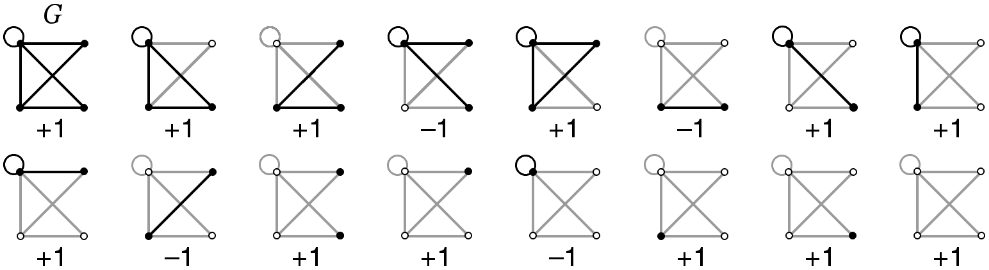


Figure 1: All vertex subgraphs of G, with the parity of the number of edges indicated.

With this perspective we are now prepared to generalize the preceding ideas. So let n be a fixed positive integer. A level n graph sum is essentially a sum over all possible vertex n-colorings of the "total edge value" of each coloring, obtained by multiplying together all the individual edge values, which depend in turn upon the colors of their endpoints. These edge values will be n^{th} roots of unity. For the sake of consistency, the value of a graph with no edges is defined to be 1.

Therefore a graph sum is determined by the particular assignment of edge values based on the coloring of their endpoints. This information is naturally organized in an $n \times n$ matrix M whose entry m_{jk} in row j, column k catalogs the n^{th} root of unity that is assigned to an edge whose endpoints have colors j and k. The matrix will be symmetric, since the order in which the endpoints are considered should not matter. For example, the graph sum introduced above is the level two graph sum given by $M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

It will be convenient to "color by number," so that a coloring θ of the vertex set $V(G)$ is technically a function $\theta : V(G) \to \{0, 1, \ldots, n-1\}$ that assigns a positive integer from 0 to $n-1$ to each vertex. We let $\mathcal{C}$ denote the set of all colorings, which contains $n^{V(G)}$ elements. By slight abuse of notation we will let $\theta(e)$ denote the pair of numbers occurring at the endpoints of an edge $e \in E(G)$, the edge set of G, so that $m_{\theta(e)}$ is the edge value of e under the assignment M for a coloring θ. With this notation the graph sum given by M may be written

$$S_M(G) = \sum_{\theta \in \mathcal{C}} \prod_{e \in E(G)} m_{\theta(e)}, \tag{2}$$

with the understanding that the empty product has a value of 1.

Notice that there is no need to restrict ourselves to simple graphs when defining graph sums; the computation may be performed just as readily if there are several edges between the same two vertices. It is natural to only consider simple graphs when dealing with level two graph sums, because edges with common endpoints will always have the same edge value; either both 1 or both -1. Hence such a pair of edges will not affect the product $\prod m_{\theta(e)}$ for any coloring. By the same token, we need only consider multigraphs with up to $n-1$ edges between any given pair of vertices when analyzing a level n graph sum.

3 Equivalent graph sums

In the sequel we will focus our attention on the case $n = 3$. As just outlined, a level three graph sum is determined by a symmetric 3×3 matrix whose entries are 1, ω or $\overline{\omega}$, where $\omega = e^{2\pi i/3}$. There are $3^6 = 729$ matrices fitting this description; our goal will be to identify, verify, and characterize those matrices which give rise to graph sums having a property analogous to our motivating level two example above. Without loss of generality, we may restrict our attention to multigraphs with at most two edges joining any pair of vertices, or joining a vertex to itself.

Clearly $S_M(G) \in \mathbb{Z}[\omega]$ for any graph sum S_M. Let $\mathcal{N}(a + b\omega) = a^2 - ab + b^2$ denote the usual algebraic norm. Then for certain matrices M, $\mathcal{N}(S_M(G))$ will always be a power of 3. For example, the graph sum described in the Introduction is given by

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{bmatrix}.$$

If G consists of two vertices with a single edge between them, then $S_M(G)$ is the sum of the entries of M, which is 3. Including a second edge between the two vertices results in $S_M(G)$ totalling the squares of the entries of M, which also yields 3. Finally, if G consists of a loop on a single vertex, then $S_M(G)$ equals the sum of the diagonal entries of M, giving $1 + 2\omega$, and $\mathcal{N}(1 + 2\omega) = 3$. We will demonstrate shortly for this particular matrix M that $\mathcal{N}(S_M(G))$ always has the form 3^m for a positive integer m in the range $|G| \leq m \leq 2|G|$.

We may catalog graph sums efficiently by observing that in many cases distinct matrices give rise to graph sums giving, for all practical purposes, the same sets of values. More precisely, we say that S_M and $S_{M'}$ are *equivalent* if $\mathcal{N}(S_M(G)) = \mathcal{N}(S_{M'}(G))$ for all multigraphs G.

Proposition 2 *Let M be a symmetric 3×3 matrix whose entries are cube roots of unity. Suppose that M' is obtained from M by permuting its rows and columns in a parallel fashion, multiplying every entry by the same cube root of unity, and possibly conjugating all entries. Then S_M and $S_{M'}$ are equivalent graph sums.*

Proof: We will show that each operation described leads to an equivalent graph sum. To begin, permuting the rows and columns of M (in a parallel fashion, so that the resulting matrix is still symmetric) simply amounts to a renumbering of the vertex colors. Since we sum over all colorings, a permutation of rows and columns will only rearrange the terms in the sum, leaving the total unchanged. Hence $S_M(G) = S_{M'}(G)$ in this case. Next, if $m'_{jk} = \omega m_{jk}$ then the product $\prod m'_{\theta(e)}$ over all edges becomes $\omega^{e(G)} \prod m_{\theta(e)}$. Hence $S_{M'}(G) = \omega^{e(G)} S_M(G)$, which does not affect the norm. Similar reasoning applies if $m'_{jk} = \overline{\omega} m_{jk}$. Finally, it is clear that if $m'_{jk} = \overline{m_{jk}}$ then $S_{M'}(G) = \overline{S_M(G)}$, which again does not alter the norm. $\square$

The operations described in proposition 2 act as the group $S_3 \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (with 36 elements)

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $(3); 81, 81, 81$ *arithmetic*	$\begin{bmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{bmatrix}$ $(6); 27, 27, 27$ *arithmetic*	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{bmatrix}$ $(18); 39, 39, 39$	$\begin{bmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega & \omega & 1 \end{bmatrix}$ $(18); 21, 21, 21$	$\begin{bmatrix} 1 & \omega & \omega \\ \omega & 1 & \overline{\omega} \\ \omega & \overline{\omega} & 1 \end{bmatrix}$ $(18); 3, 3, 3$	$\begin{bmatrix} 1 & 1 & \omega \\ 1 & 1 & \overline{\omega} \\ \omega & \overline{\omega} & 1 \end{bmatrix}$ $(18); 9, 9, 9$	
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \omega \end{bmatrix}$ $(18); 57, 21, 3$	$\begin{bmatrix} 1 & \omega & 1 \\ \omega & 1 & 1 \\ 1 & 1 & \omega \end{bmatrix}$ $(18); 27, 27, 9$ *arithmetic*	$\begin{bmatrix} 1 & \overline{\omega} & 1 \\ \overline{\omega} & 1 & 1 \\ 1 & 1 & \omega \end{bmatrix}$ $(18); 21, 3, 3$	$\begin{bmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega & \omega & \omega \end{bmatrix}$ $(18); 21, 21, 21$	$\begin{bmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & \omega \end{bmatrix}$ $(18); 39, 3, 3$	$\begin{bmatrix} 1 & \overline{\omega} & \omega \\ \overline{\omega} & 1 & \omega \\ \omega & \omega & \omega \end{bmatrix}$ $(18); 9, 27, 9$ *arithmetic*	
$\begin{bmatrix} 1 & 1 & \overline{\omega} \\ 1 & 1 & \overline{\omega} \\ \overline{\omega} & \overline{\omega} & \omega \end{bmatrix}$ $(18); 9, 81, 27$ *arithmetic*	$\begin{bmatrix} 1 & \omega & \overline{\omega} \\ \omega & 1 & \overline{\omega} \\ \overline{\omega} & \overline{\omega} & \omega \end{bmatrix}$ $(18); 3, 39, 21$	$\begin{bmatrix} 1 & \overline{\omega} & \overline{\omega} \\ \overline{\omega} & 1 & \overline{\omega} \\ \overline{\omega} & \overline{\omega} & \omega \end{bmatrix}$ $(18); 21, 39, 3$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \omega \\ 1 & \omega & \omega \end{bmatrix}$ $(36); 27, 9, 9$	$\begin{bmatrix} 1 & \omega & 1 \\ \omega & 1 & \omega \\ 1 & \omega & \omega \end{bmatrix}$ $(36); 21, 3, 3$	$\begin{bmatrix} 1 & \overline{\omega} & 1 \\ \overline{\omega} & 1 & \omega \\ 1 & \omega & \omega \end{bmatrix}$ $(36); 3, 3, 3$	
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{bmatrix}$ $(36); 21, 39, 12$	$\begin{bmatrix} 1 & \omega & 1 \\ \omega & 1 & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{bmatrix}$ $(36); 3, 21, 12$	$\begin{bmatrix} 1 & \overline{\omega} & 1 \\ \overline{\omega} & 1 & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{bmatrix}$ $(36); 9, 9, 0$	$\begin{bmatrix} 1 & 1 & \omega \\ 1 & 1 & \overline{\omega} \\ \omega & \overline{\omega} & \omega \end{bmatrix}$ $(36); 3, 3, 21$	$\begin{bmatrix} 1 & \omega & \omega \\ \omega & 1 & \overline{\omega} \\ \omega & \overline{\omega} & \omega \end{bmatrix}$ $(36); 9, 9, 9$	$\begin{bmatrix} 1 & \overline{\omega} & \omega \\ \overline{\omega} & 1 & \overline{\omega} \\ \omega & \overline{\omega} & \omega \end{bmatrix}$ $(36); 3, 21, 3$	
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \overline{\omega} \end{bmatrix}$ $(18); 36, 9, 0$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega \\ 1 & \omega & \overline{\omega} \end{bmatrix}$ $(36); 12, 3, 3$	$\begin{bmatrix} 1 & 1 & \omega \\ 1 & \omega & 1 \\ \omega & 1 & \overline{\omega} \end{bmatrix}$ $(36); 12, 39, 3$	$\begin{bmatrix} 1 & \omega & 1 \\ \omega & \omega & 1 \\ 1 & 1 & \overline{\omega} \end{bmatrix}$ $(36); 12, 21, 12$	$\begin{bmatrix} 1 & 1 & \omega \\ 1 & \omega & \overline{\omega} \\ \omega & \overline{\omega} & \overline{\omega} \end{bmatrix}$ $(18); 0, 9, 0$	$\begin{bmatrix} 1 & 1 & \overline{\omega} \\ 1 & \omega & \omega \\ \overline{\omega} & \omega & \overline{\omega} \end{bmatrix}$ $(12); 0, 27, 9$ *arithmetic*	$\begin{bmatrix} 1 & \overline{\omega} & \omega \\ \overline{\omega} & \omega & 1 \\ \omega & 1 & \overline{\omega} \end{bmatrix}$ $(6); 0, 81, 27$ *arithmetic*

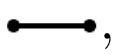
Table 1: Representatives of the thirty-one distinct level three graph sums, with number of matrices per equivalence class in parentheses and $\mathcal{N}(S_M(G))$ shown for three particular graphs.

on the set of 729 graph sum matrices. Hence the "lemma that is not Burnside's" may be applied to count the number of equivalence classes of matrices. We omit the details; the calculation reduces to computing

$$\frac{1}{36}(1 \cdot 729 + 3 \cdot 81 + 15 \cdot 9 + 9 \cdot 1 + 8 \cdot 0) = 31.$$

A careful search yields all thirty-one equivalence classes of matrices; a representative from each is listed in Table 1 along with the number of matrices in that equivalence class, in parentheses. The matrices are partitioned into three sets according to the composition of the main diagonal. It remains to show these matrices actually yield nonequivalent graphs sums.

Proposition 3 *There are exactly thirty-one distinct (nonequivalent) level three graph sums.*

Proof: We have already seen that there are at most thirty-one different level three graph sums. Observe that when G consists of a loop on a single vertex, $\mathcal{N}(S_M(G))$ depends only the sum of the diagonal entries of M. Hence the norm will equal 9, 3, and 0, respectively, for matrices in the top, middle, and bottom sets of matrices in Table 1. To further distinguish between matrices in each section, we compute $\mathcal{N}(S_M(G))$ for the graphs , , and . These values are displayed beneath the corresponding matrix, separated by commas. The only two matrices with identical values are the third matrix in the second row and the fifth matrix in the third row. However, $\mathcal{N}(S_M(G))$ equals 21 and 12 for these two graph sums when $G =$ . Therefore all thirty-one equivalence classes of matrices yield nonequivalent graphs sums, as claimed. □

4 Arithmetic graph sums

We are interested in identifying those graph sums with a property analogous to the level two example presented earlier. Therefore we say that a level three graph sum S_M is *arithmetic* if $\mathcal{N}(S_M(G)) = 0$ or 3^m, where m is a positive integer, for all multigraphs G. Returning to the norms computed in Table 1, we see that there are fourteen candidates for level three arithmetic graph sums. Upon further inspection, half of these persist in giving values with the desired norms and are labelled 'arithmetic,' while the others fail to do so. For example, if G is the multigraph on vertices v_1, v_2, and v_3 having edges v_1v_2, v_1v_3, v_2v_3, v_2v_3 and single loops at v_1 and v_2, then $\mathcal{N}(S_M(G))$ takes on the values 171, 225, 117, 117, 63, 117, and 36 for the other seven matrices, in the order that they appear in Table 1.

We postpone the proofs that the seven indicated matrices do yield arithmetic graph sums, since it will be helpful to first identify common features of these matrices. By totalling the size of the equivalence classes of the matrices marked as arithmetic, we discover that of the 729 level three graphs sums, precisely 81 of them are arithmetic. The fact that this figure is also a power of 3 suggests that there is a unifying characterization of matrices giving rise to arithmetic graphs sums. We present three possibilities here.

Proposition 4 *Let M be a 3×3 symmetric matrix whose entries are cube roots of unity. Then S_M is an arithmetic graph sum if and only if the product of the entries along each row is constant.*

Proof: It is clear that permuting the rows and columns of M will not affect whether or not the row products are constant. The same is true of multiplying M by a scalar or conjugating the entries of M. Hence a constant row product is a property of the entire equivalence class of matrices. By inspection, the matrices in Table 1 all of whose rows have the same product are exactly those marked as arithmetic. □

Note that the set W of all symmetric 3×3 matrices whose entries are cube roots of unity is an abelian group with respect to entrywise multiplication. The previous proposition then implies that the matrices giving arithmetic graph sums form a subgroup $A \subset W$. However, the group structure does not factor through the equivalence relation, since this would yield a group of order thirty-one having elements of order three.

We mention a second criterion in passing because it is somewhat unexpected; the extent to which it is significant or can be generalized to higher level graph sums is unclear. This criterion demonstrates, for level three at least, that the determinant may also be used to distinguish between arithmetic and non-arithmetic graph sums.

Proposition 5 *Let* $M \in W$ *be a* 3×3 *symmetric matrix whose entries are cube roots of unity. If* $\mathcal{N}(\det(M)) = 9$ *or* $\operatorname{rank}(M) = 1$ *then* S_M *is an arithmetic graph sum, while if* $\mathcal{N}(\det(M)) = 3$ *or* $\operatorname{rank}(M) = 2$ *then* S_M *is not arithmetic.*

Proof: We proceed as before. Observe that $\mathcal{N}(\det(M))$ and $\operatorname{rank}(M)$ are both fixed by a permutation of the rows and columns of M, by scalar multiplication by a cube root of unity, and by conjugation. Hence we need only check the assertion on the representatives listed in Table 1. (In particular, the only possible values for $\mathcal{N}(\det(M))$ are 9, 3, and 0.) The verification is straightforward, so we omit the details. $\square$

The final characterization we discuss is also the most powerful. Let $f \in \mathbb{Z}[x, y]$ be a polynomial in two variables with integer coefficients. If $f(x, y)$ is symmetric, then setting $m_{jk} = \omega^{f(j,k)}$ defines a symmetric matrix whose entries are cube roots of unity, which we denote by M_f. Since $\omega^3 = 1$ we may just as well (and it will be advantageous to) consider the coefficients of $f(x, y)$ modulo 3. We will show that such polynomials with $\deg(f) \leq 2$ lead to precisely the arithmetic graph sums.

Proposition 6 *Let* B *be the set of all symmetric polynomials in* x *and* y *having degree two or less whose coefficients are 0, 1, or 2. Then* S_M *is an arithmetic graph sum if and only if* $M = M_f$ *for some* $f \in B$*.*

Proof: Polynomials in B all have the form

$$f(x, y) = a(x^2 + y^2) + bxy + c(x + y) + d, \qquad a, b, c, d \in \{0, 1, 2\}. \tag{3}$$

Note that B is an abelian group with 81 elements with respect to addition mod 3. This group has generators $x^2 + y^2$, xy, $x + y$, and 1, each of order three. Clearly the map $\phi : B \to W$ mapping $f \mapsto M_f$ is a homomorphism; it suffices to show that this map is injective with image contained in A, the subgroup of matrices giving arithmetic graph sums. If $f \in \ker(\phi)$ then $f(x, y) \equiv 0 \bmod 3$ for all x and y. This leads to six congruences mod 3.

$$d \equiv 0, \quad a + c + d \equiv 0, \quad a + 2c + d \equiv 0, \quad 2a + b + 2c + d \equiv 0, \quad 2a + 2b + d \equiv 0, \quad 2a + b + c + d \equiv 0.$$

The only solution is $a \equiv b \equiv c \equiv d \equiv 0 \bmod 3$, hence $\ker(\phi)$ is trivial, so the map is injective.

$$\begin{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & \omega & \overline{\omega} \\ \omega & \overline{\omega} & 1 \\ \overline{\omega} & 1 & \omega \end{bmatrix} & \begin{bmatrix} 1 & \omega & \omega \\ \omega & \overline{\omega} & \overline{\omega} \\ \omega & \overline{\omega} & \overline{\omega} \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{bmatrix} & \begin{bmatrix} 1 & \omega & \omega \\ \omega & \omega & 1 \\ \omega & 1 & \omega \end{bmatrix} \\ f_1 = 0 & f_2 = x + y & f_3 = x^2 + y^2 & f_4 = xy & f_5 = x^2 + 2xy + y^2 \end{matrix}$$

$$\begin{matrix} \begin{bmatrix} 1 & \omega & \omega \\ \omega & 1 & \omega \\ \omega & \omega & 1 \end{bmatrix} & \begin{bmatrix} 1 & \overline{\omega} & 1 \\ \overline{\omega} & \overline{\omega} & \omega \\ 1 & \omega & \omega \end{bmatrix} \\ f_6 = x^2 + xy + y^2 & f_7 = x^2 + xy + y^2 + x + y \end{matrix}$$

Table 2: Representatives of the seven arithmetic graph sums, with generating polynomials.

When writing down a matrix M_f it will be more natural to use indices j and k in the range $0 \leq j, k \leq 2$ (rather than $1 \leq j, k \leq 3$) for the entries m_{jk} of M. With this convention the images of the generators $x^2 + y^2$, xy, $x + y$, and 1 are the matrices

$$\begin{bmatrix} 1 & \omega & \omega \\ \omega & \overline{\omega} & \overline{\omega} \\ \omega & \overline{\omega} & \overline{\omega} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{bmatrix}, \quad \begin{bmatrix} 1 & \omega & \overline{\omega} \\ \omega & \overline{\omega} & 1 \\ \overline{\omega} & 1 & \omega \end{bmatrix}, \quad \begin{bmatrix} \omega & \omega & \omega \\ \omega & \omega & \omega \\ \omega & \omega & \omega \end{bmatrix}.$$

These four are equivalent to the fifth, fourth, seventh, and first of the matrices marked 'arithmetic' in Table 1. Hence the image of ϕ is contain within the subgroup A. But $|A| = 81$ and ϕ is injective, hence the image must be equal to A. Therefore the matrices M_f for $f \in B$ correspond to the arithmetic graph sums in a one-to-one fashion. This completes the proof. $\square$

We are now able to present the seven distinct level three graph sums identified as arithmetic in a way that will greatly facilitate their proof. In Table 2 we display these matrices along with their corresponding polynomials, listed more or less in order of increasing complexity. In each case we have chosen the simplest available polynomial, so in general these matrices will not match those appearing in Table 1, although each is equivalent to a different arithmetic matrix in that figure. To ease notation, we will refer to the corresponding matrices and graph sums as M_1 through M_7 and S_1 through S_7.

5 Proof

We now return to the task of demonstrating that the seven equivalence classes labelled 'arithmetic' in Table 1 actually live up to their billing. Prior to now we have not been concerned with the number of edges (or loops) joining a given pair of vertices, other than to observe that we may take this number to be 0, 1, or 2 without loss of generality. For the remainder of this section, though, we will always regard these numbers modulo 3. (This is permissible because edge or loop counts will only appear in an exponent of ω.) For example, including two additional edges between a pair of vertices currently joined by a single edge results in no edges. These considerations will apply to any graph property involving the presence or absence of edges, such as determining connected components or discussing disjoint graphs.

We first establish an important multiplicative property of graph sums.

Lemma 7 *Let G_1 and G_2 be disjoint graphs. Then*

$$S_M(G_1 \cup G_2) = S_M(G_1) \cdot S_M(G_2) \tag{4}$$

for any graph sum S_M.

Proof: We will assume that neither G_1 nor G_2 is empty, but for the sake of consistency we also define $S_M(\emptyset) = 1$. Writing $G = G_1 \cup G_2$, observe that a three-coloring $\theta \in \mathcal{C}$ of $V(G)$ corresponds to three-colorings $\theta_1 \in \mathcal{C}_1$ and $\theta_2 \in \mathcal{C}_2$ of $V(G_1)$ and $V(G_2)$. Furthermore, the product over all edges $e \in E(G)$ factors into products over $e \in E(G_1)$ and $e \in E(G_2)$, since there are no edges between $V(G_1)$ and $V(G_2)$. Thus for any graph sum S_M we have

$$\begin{aligned}
S_M(G) &= \sum_{\theta \in \mathcal{C}} \prod_{e \in E(G)} m_{\theta(e)} \\
&= \sum_{\theta_1 \in \mathcal{C}_1} \sum_{\theta_2 \in \mathcal{C}_2} \left(\prod_{e \in E(G_1)} m_{\theta_1(e)} \cdot \prod_{e \in E(G_2)} m_{\theta_2(e)} \right) \\
&= \left(\sum_{\theta_1 \in \mathcal{C}_1} \prod_{e \in E(G_1)} m_{\theta_1(e)} \right) \left(\sum_{\theta_2 \in \mathcal{C}_2} \prod_{e \in E(G_2)} m_{\theta_2(e)} \right) \\
&= S_M(G_1) \cdot S_M(G_2).
\end{aligned}$$

This completes the proof. □

Thus far we have seen that the only candidates for arithmetic graph sums are those given by the 81 matrices in the subgroup $A \subset W$, which fall into seven equivalence classes. A representative matrix from each class appears in Table 2, each generated by a certain symmetric polynomial in $\mathbb{Z}[x, y]$. We will now demonstrate that each of these matrices M_1 to M_7 does in fact yield an arithmetic graph sum, which we are calling S_1 through S_7. We begin by showing that graph sums S_1, S_2, and S_3 are arithmetic.

Since every entry of M_1 is 1, the value of $S_1(G)$ depends only on the number of vertices of G, not on which edges occur. We have

$$S_1(G) = \sum_{\theta \in \mathcal{C}} \prod_{e \in E(G)} m_{\theta(e)} = \sum_{\theta \in \mathcal{C}} 1 = 3^{|G|}. \tag{5}$$

Thus $S_1(G)$ "counts" the number of vertices of G.

Graph sum S_2 is relatively simple to analyze because the coefficient of xy in f_2 is 0. This has the effect of making $\operatorname{rank}(M_2) = 1$. As we shall see, this will allow us to construct a graph G' with $S_2(G') = S_2(G)$, but such that G' has no edges (only loops). We can then invoke the multiplicative property (4) to conclude that S_2 is arithmetic.

If G has no edges then we may take $G' = G$ and skip to the next step. Otherwise, suppose that vertices u_1 and u_2 are joined by at least one edge. We claim that the term in the graph sum corresponding to a particular coloring θ will be unaltered by eliminating one of these edges and including two additional loops at both of vertices u_1 and u_2. If these vertices have colors q_1 and q_2, then an edge between them will contribute $\omega^{q_1+q_2}$ to the product $\prod m_{\theta(e)}$, keeping in mind that $f_2(x, y) = x + y$. On the other hand, the additional loops will contribute

$$(\omega^{q_1+q_1})^2 \cdot (\omega^{q_2+q_2})^2 = \omega^{4q_1+4q_2} = \omega^{q_1+q_2},$$

which exactly compensates for the missing edge. Since this holds for any coloring θ, the value of the graph sum will be unchanged.

By performing this operation as long as edges remain in our graph we obtain a graph G' having no edges, with $S_2(G) = S_2(G')$. But it is straight-forward to verify that if H is a graph consisting of loops at a single vertex then $S_2(H) = 0$ or 3. Hence $\mathcal{N}(S_2(H)) = 0$ or 9. Invoking (4), we conclude that $\mathcal{N}(S_2(G)) = 0$ or 3^k for some positive (even) integer k, so S_2 is arithmetic.

The reasoning for S_3 is entirely analogous; one need only replace each occurrence of q_1 or q_2 by q_1^2 or q_2^2, respectively, in the above argument. We again conclude that there exists a graph G' with no edges for which $S_3(G) = S_3(G')$. In this case we find that $S_3(H) = 1 + 2\omega$, $1 + 2\overline{\omega}$, or 3 when $|H| = 1$. Hence $\mathcal{N}(S_2(H)) = 3$ or 9. By (4) we conclude that $\mathcal{N}(S_2(G)) = 3^k$ for some positive integer k, so S_3 is also arithmetic.

There is a combinatorial interpretation of $S_2(G)$ and $S_3(G)$, just as there was for $S_1(G)$. Note that the degree of each vertex is invariant, modulo 3, under the process described above of replacing edges by additional loops at their ends. But $S_2(H) = 0$ for a graph H with a single vertex v unless $\deg(v) \equiv 0 \bmod 3$. Hence $S_2(G)$ is a sort of characteristic function, equalling 0 unless the degree of every vertex of G is divisible by 3, in which case $S_2(G) = 3^{|G|}$. Similarly, $\mathcal{N}(S_3(H)) = 3$ unless $\deg(v) \equiv 0 \bmod 3$, when $\mathcal{N}(S_3(H)) = 9$. Hence $\mathcal{N}(S_3(G))$ "counts" the number of such vertices, in that $\mathcal{N}(S_3(G)) = 3^{|G|+t}$, where t is the number of vertices of G whose degree is a multiple of 3.

To handle S_4 we begin as before by eliminating edges wherever possible. Suppose that G contains a vertex u with one or two loops; we will demonstrate that we may eliminate all edges at u. Consider any other vertex v joined to u by edges. To avoid introducing an overabundance of variables we will take two edges between u and v and one loop at u to illustrate the argument. We will also assume that $|G| \geq 3$ to ensure nonempty sums and products below. (By omitting these sums and products, the same argument gives the result when $|G| = 2$.)

Label the remaining vertices of G as v_1 through v_n, where v_k is joined to u by r_k edges, as

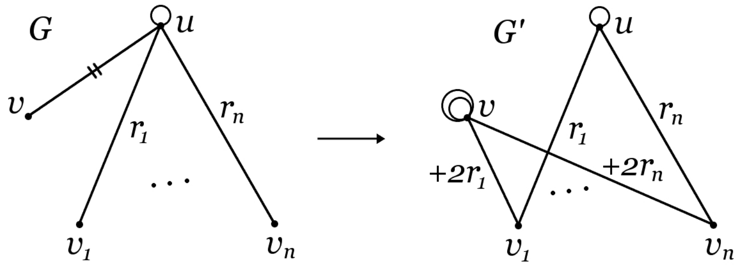


Figure 2: Isolating a vertex u with loops to show that S_4 is arithmetic.

shown in Fig. 2. Consider the three colorings in which u has color q for $q = 0, 1, 2$ while v has fixed color a and v_k has color a_k for $1 \le k \le n$. Since $f_4(x, y) = xy$, the contribution of these three colorings to the overall graph sum is

$$\omega^h \sum_{q=0}^{2} \left(\omega^{q^2} \omega^{2aq} \prod_{k=1}^{n} (\omega^{a_k q})^{r_k} \right), \tag{6}$$

where h depends only on loops and edges not at u. The exponent of ω within the sum is given by $q^2 + 2aq + \sum r_k a_k q$. (An undelimited sum $\sum$ will always mean $\sum_{k=1}^{n}$.) Note that the $2aq$ term arises due to the edges between u and v. To eliminate it algebraically, we replace q by $q + 2a$, which does not change the value of $S_4(G)$ since we are summing over q and treating quantities mod 3. We obtain

$$(q + 2a)^2 + 2a(q + 2a) + \sum_{k=1}^{n} r_k a_k (q + 2a) \equiv q^2 + 2a^2 + \sum_{k=1}^{n} r_k a_k q + \sum_{k=1}^{n} 2 r_k a a_k \mod 3. \tag{7}$$

But the right-hand side gives precisely the exponent of ω that would appear if we were to eliminate the two edges between u and v, include two additional loops at v, and also increase the number of edges between v and v_k by $2r_k$ for $1 \le k \le n$. This operation is illustrated in Fig. 2. In a rather delightful manner, the algebraic separation of a and q informs the graphical separation of u and v that preserves the value of the graph sum.

In the same way one finds in general that if there are l loops at u and r edges between u and v (with $l, r \not\equiv 0 \bmod 3$), then eliminating these edges may be exactly compensated for by including $2l$ additional loops at v and increasing the number of edges between v and v_k by lrr_k. Since this process does not alter the loops or edges at u (except to v), we may perform it repeatedly until u is no longer connected to the remainder of the graph. Furthermore, observe that this process does not introduce new edges to isolated vertices, so we may continue to separate vertices with loops until none remain connected by edges to the rest of the graph.

At this point we may assume that the only remaining edges join vertices without loops. Let u_1 and u_2 be two such vertices; suppose they are joined by a single edge. If this edge is not already

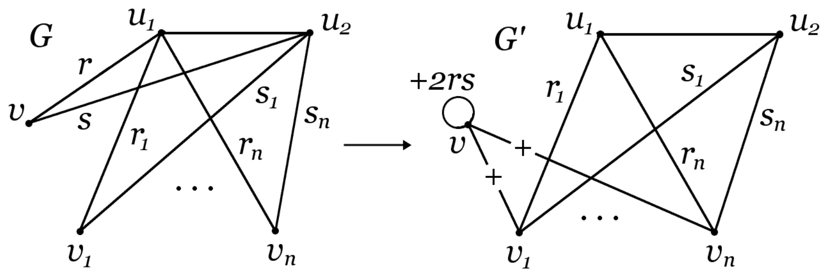


Figure 3: Isolating an edge u_1u_2 without loops to show that S_4 is arithmetic.

a connected component, then we may modify the graph to isolate it without changing the value of the graph sum as follows. Label the remaining vertices of G as v, v_1, $\ldots$, v_n, and let u_1 and u_2 be joined to v_k (resp. v) by r_k and s_k (resp. r and s) edges, as in Fig. 3. (As before, simply omit the sums over k if $|G| = 3$.) Consider the nine colorings of G in which v has fixed color a, v_k has color a_k, but the colors q_1 and q_2 of u_1 and u_2 vary. These colorings contribute

$$\omega^h \sum_{q_1,q_2=0}^{2} \left(\omega^{q_1q_2+raq_1+saq_2} \prod_{k=1}^{n} \omega^{r_ka_kq_1+s_ka_kq_2} \right), \tag{8}$$

where h depends on edges and loops not at u_1 or u_2. Replacing q_1 by $q_1 + 2as$ and q_2 by $q_2 + 2ar$ does not affect the overall sum mod 3, but changes the exponent of ω within the sum to

$$q_1q_2 + 2rsa^2 + \left(\sum_{k=1}^{n} r_ka_kq_1 + s_ka_kq_2 \right) + \sum_{k=1}^{n} (2r_ks + rs_k)aa_k, \tag{9}$$

reducing mod 3 where possible. The variables q_1 and q_2 are now separate from a. We discover that taking out the edges from v to u_1 or u_2 and replacing them with $2rs$ additional loops at v and $(2r_ks + 2rs_k)$ more edges between v and v_k for $1 \leq k \leq n$ leaves the graph sum unchanged. As before, we may repeat this process until edge u_1u_2 is isolated. Finally, one readily checks that for two edges between u_1 and u_2 the same operation accomplishes the isolation if we instead use rs additional loops at v and $(r_ks + rs_k)$ more edges between v and v_k.

It is interesting to note that neither operation alone suffices to reduce an arbitrary graph G. But by implementing them together we may always obtain a graph G' with $S_4(G) = S_4(G')$ whose components consist of either a single vertex with 0, 1, or 2 loops, a single edge without loops, or a double edge without loops. The value of S_4 on these five graphs is 3, $1 + 2\omega$, $1 + 2\overline{\omega}$, 3, and 3, having norms of 9, 3, 3, 9, and 9, respectively. We deduce that $\mathcal{N}(S_4(G)) = 3^m$ for $|G| \leq m \leq 2|G|$. Looking more closely, we see that each vertex contributes a single factor of 3 to $\mathcal{N}(S_4(G))$ except for single vertices without loops. Hence the number t of such vertices will be the same regardless of how G is reduced to G', and $\mathcal{N}(S_4(G)) = 3^{|G|+t}$.

The demonstration that S_5 is arithmetic involves more algebra since $f_5(x,y) = (x+y)^2$ has both x^2+y^2 and xy terms. Let u be a vertex having l loops, joined to the remaining vertices v, $v_1, \ldots, v_n$ of G by $r, r_1, \ldots, r_n$ edges, respectively. Define $d = l + r + r_1 + \cdots + r_n$; thus d is not quite the degree of u. If $d \not\equiv 0 \bmod 3$ then it is possible to isolate vertex u without altering the graph sum. Assuming that $r \not\equiv 0 \bmod 3$ already, we are able to remove the edges between u and v as follows. If we assign colors $q, a, a_1, \ldots, a_k$ to $u, v, v_1, \ldots\ a_k$ and sum over q as usual, we see that the edges and loops at u lead to an exponent for ω of

$$l(2q)^2 + r(q+a)^2 + \sum_{k=1}^{n} r_k(q+a_k)^2, \tag{10}$$

Note that the coefficient of q^2 will be d (working mod 3), and since $d \not\equiv 0 \bmod 3$ we can eliminate the cross term aq with a suitable substitution. We use $q \mapsto q + 2rda$, which does not affect the sum over q but transforms the exponent of ω (after a fair amount of algebra) into

$$(l+r)q^2 + (d + 2r + 2lrd)a^2 + \sum_{k=1}^{n} r_k \left((q+a_k)^2 + rda_k^2 + 2rd(a+a_k)^2\right). \tag{11}$$

This corresponds to eliminating the r edges between u and v, including r additional loops at u, $(d+2r+2lrd)$ more loops at v, rr_kd extra loops at v_k, and another $2rr_kd$ edges between v and v_k. The exact algebraic expressions are not as important, though, as the fact that such an identity exists. We do note, however, that the value of d remains fixed by this process, as do the number of edges from u to v_k. Hence we may perform this operation repeatedly to isolate vertex u.

If there are no edges (joining distinct vertices) left after applying the above process wherever possible then the graph is completely disconnected and we may omit the following step. Otherwise, suppose that there are $e \not\equiv 0 \bmod 3$ edges joining u_1 and u_2, having l_1 and l_2 loops. Label the remaining vertices of G as $v, v_1, \ldots, v_n$, and let u_1 and u_2 be joined to v_k (resp. v) by r_k and s_k (resp. r and s) edges, similar to Fig. 3. Assigning colors $q_1, q_2, a, a_1, \ldots, a_n$ to these vertices as usual and summing over q_1 and q_2 leads to an expression of the form $\omega^h \sum_{q_1,q_2} \omega^{h(q_1,q_2)}$, where the exponent of ω (mod 3) is given by

$$l_1q_1^2 + l_2q_2^2 + e(q_1+q_2)^2 + r(q_1+a)^2 + s(q_2+a)^2 + \sum_{k=1}^{n} \Big(r_k(q_1+a_k)^2 + s_k(q_2+a_k)^2\Big). \tag{12}$$

Since we are assuming that $d_1 = l_1 + e + r + \sum r_k \equiv 0 \bmod 3$ and $d_2 = l_2 + e + s + \sum s_k \equiv 0 \bmod 3$, the above expression reduces to

$$2eq_1q_2 + 2raq_1 + 2sq_2a + ra^2 + sa^2 + \sum_{k=1}^{n} \Big(2r_kq_1a_k + 2s_kq_2a_k + r_ka_k^2 + s_ka_k^2\Big). \tag{13}$$

Making the substitutions $q_1 \mapsto q_1 - esa$ and $q_2 \mapsto q_2 - era$ will algebraically eliminate the cross-terms involving aq_1 and aq_2, which will correspond to removing the edges from v to u_1 and u_2. This step entails the most algebra needed for any of the arithmetic graph sums; the resulting expression may be written (mod 3) as

$$e(q_1+q_2)^2+(l_1+r)q_1^2+(l_2+s)q_2^2+\sum_{k=1}^{n}\Big(r_k(q_1+a_k)^2+s_k(q_2+a_k)^2\Big)+ \tag{14}$$
$$(r+s+ers)a^2+\sum_{k=1}^{n}(er_ks+ers_k)a^2+(2er_ks+2ers_k)(a+a_k)^2+(er_ks+ers_k)a_k^2.$$

We have opted for compactness over clarity by writing a^2 instead of $(a+a)^2$ mod 3. We conclude that the graph sum will be unchanged if we remove the $r+s$ edges from v to u_1 and u_2 and include r more loops at u_1, s more loops at u_2, add $r+s+ers+\sum e(r_ks+rs_k)$ more loops at v, insert $2e(r_ks+rs_k)$ more edges from v to v_k, and include $e(r_ks+rs_k)$ more loops at v_k. This operation leaves the value of d_1 and d_2 at vertices u_1 and u_2 unchanged mod 3, so we may repeat it until edges from u_1 to u_2 are isolated.

In this manner we eventually obtain a graph G' with $S_5(G)=S_5(G')$ whose components consist of either a single vertex with 0, 1, or 2 loops, a single edge with double loops at each vertex, or a double edge with single loops. The value of S_5 on these five graphs is 3, $1+2\omega$, $1+2\overline{\omega}$, 3, and 3, having norms of 9, 3, 3, 9, and 9, respectively. (Observe that the situation is identical to the case of S_4.) We deduce that $\mathcal{N}(S_5(G))=3^m$ for $|G|\le m\le 2|G|$. As before, we see that each vertex contributes a single factor of 3 to $\mathcal{N}(S_5(G))$ except for single vertices without loops. Hence the number t of such vertices will be the same regardless of how G is reduced to G', and $\mathcal{N}(S_5(G))=3^{|G|+t}$.

The last two graph sums are given by polynomials with the same quadratic terms, exhibit comparable behavior, and may be proven to be arithmetic in a similar fashion. We will show that $\mathcal{N}(S_6(G))=3^m$ while $\mathcal{N}(S_7(G))=3^m$ or 0, for $|G|+1\le m\le 2|G|$ in either case. Note that because of these bounds on m the strategy used previously cannot be employed here. For example, if G is the graph on five vertices having a single edge between each pair of vertices except for one instance of a pair of parallel edges, then $\mathcal{N}(S_6(G))=3^6$. If we could find a graph G' on the same vertex set such that $S_6(G)=S_6(G')$ with $G'=G_1\cup G_2$ as a disjoint union, then we would have

$$\mathcal{N}(S_6(G))=\mathcal{N}(S_6(G_1))\cdot\mathcal{N}(S_6(G_2))=3^{m_1+m_2},$$

where $m_1+m_2\ge(|G_1|+1)+(|G_2|+1)=7$, yielding a contradiction.

The sixth graph sum we consider has one of the nicest proofs. We write $f_6(x,y)=x^2+xy+y^2$ or $f_6(x,y)=(x-y)^2$, depending upon which is more convenient, since they are congruent mod 3. To begin, note that the presence of loops at vertices of G is irrelevant, since the diagonal entries of M_6 are all equal to 1. Hence we may assume without loss of generality that G has no loops. Next suppose that some vertex u of G has degree $d\not\equiv 0$ mod 3. Label the remaining vertices (of which there must be at least one) as $v_1,\ldots,v_n$, where v_k is joined to u by r_k edges and has fixed color a_k, so that $d = \sum r_k$. If u has color $q = 0, 1, 2$, these colorings contribute

$$\omega^h \sum_{q=0}^{2} \prod_{k=1}^{n} \omega^{r_k(q^2+a_kq+a_k^2)} \tag{15}$$

to the graph sum, where h depends only on edges not at u. The exponent of ω within the sum is

$$dq^2 + \left(\sum_{k=1}^{n} r_k a_k\right) q + \sum_{k=1}^{n} r_k a_k^2. \tag{16}$$

Making the substitution $q \mapsto q + d\sum r_k a_k$ leaves the sum over q unaltered as usual, but will eliminate the linear term working mod 3. The above expression becomes

$$dq^2 + 2d\left(\sum_{k=1}^{n} r_k a_k\right)^2 + \sum_{k=1}^{n} r_k a_k^2, \tag{17}$$

since $d^2 \equiv 1 \bmod 3$ when $d \not\equiv 0 \bmod 3$. Factoring out ω^{dq^2} and summing over q gives $(1 + 2\omega^d)$, which has norm 3 whether $d \equiv 1$ or 2. The rest of the exponent may be rewritten in a beautiful manner as a sum of squares. Taking our cue from the cross terms $a_j a_k$ arising from the middle term of (17) we have

$$2d\left(\sum_{k=1}^{n} r_k a_k\right)^2 + \sum_{k=1}^{n} r_k a_k^2 \;=\; 4d\sum_{j<k} r_j r_k (a_j - a_k)^2 + \sum_{k=1}^{n} a_k^2 \left(r_k + 2dr_k^2 - 4d\sum_{j\neq k} r_j r_k\right). \tag{18}$$

But the final expression within parentheses reduces mod 3 to

$$r_k + 2dr_k^2 + 2d\sum_{j\neq k} r_j r_k \equiv r_k + 2dr_k \sum_{k=1}^{n} r_k \equiv r_k + 2d^2 r_k \equiv 0 \bmod 3. \tag{19}$$

Therefore we conclude that

$$\omega^h \sum_{q=0}^{2} \prod_{k=1}^{n} \omega^{r_k(q^2+a_kq+a_k^2)} = \omega^h (1 + 2\omega^d) \prod_{j<k} \omega^{dr_j r_k (a_j - a_k)^2}. \tag{20}$$

for any coloring $a_1, \ldots, a_n$ of $v_1, \ldots, v_n$. Hence if we create a graph G' by deleting vertex u and all edges at u from G and including $dr_j r_k$ more edges between v_j and v_k for $1 \leq j < k \leq n$, then $S_6(G) = (1 + 2\omega^d) S_6(G')$. In particular, $\mathcal{N}(S_6(G)) = 3\mathcal{N}(S_6(G'))$.

On the other hand, if the degree of every vertex of G is divisible by 3, then suppose that G contains edges, say $r \not\equiv 0 \bmod 3$ edges between u_1 and u_2. Label the remaining vertices of G as $v_1, \ldots, v_n$, where v_k is joined to u_1 and u_2 by r_k and s_k edges, so that $r + \sum r_k \equiv r + \sum s_k \equiv 0 \bmod 3$. If v_k has fixed color a_k, while u_1 and u_2 have colors $q_1, q_2 = 0, 1, 2$, these nine colorings contribute

$$\omega^h \sum_{q_1,q_2=0}^{2} \left(\omega^{r(q_1-q_2)^2} \prod_{k=1}^{n} \omega^{r_k(q_1-a_k)^2} \cdot \prod_{k=1}^{n} \omega^{s_k(q_2-a_k)^2}\right). \tag{21}$$

to the graph sum. Simplifying the exponent of ω mod 3 yields

$$rq_1q_2 + \left(\sum_{k=1}^{n} r_k a_k\right) q_1 + \left(\sum_{k=1}^{n} s_k a_k\right) q_2 + \sum_{k=1}^{n} (r_k + s_k) a_k^2. \tag{22}$$

We then replace q_1 and q_2 by $q_1 - r\sum s_k a_k$ and $q_2 - r\sum r_k a_k$ to eliminate the linear terms. Since $r^2 \equiv 1 \bmod 3$ the resulting expression becomes

$$rq_1q_2 - r\left(\sum_{k=1}^{n} r_k a_k\right)\left(\sum_{k=1}^{n} s_k a_k\right) + \sum_{k=1}^{n} (r_k + s_k) a_k^2. \tag{23}$$

The sum over q_1 and q_2 of $\omega^{rq_1q_2}$ is 3 whether $r \equiv 1$ or 2. And just as before, the remaining terms resolve nicely into a sum of squares

$$\sum_{j<k} 2r(r_j s_k + r_k s_j)(a_j - a_k)^2 + \sum_{k=1}^{n} a_k^2 \left(2rr_k s_k + r_k + s_k - \sum_{j \neq k} 2r(r_j s_k + r_k s_j)\right), \tag{24}$$

since the coefficient of a_k^2 reduces to $0 \bmod 3$, as the reader may confirm. Hence if we create a graph G' by deleting vertices u_1 and u_2 from G, along with all edges at these vertices, and including $2r(r_j s_k + r_k s_j)$ more edges between v_j and v_k for $1 \leq j < k \leq n$, then $S_6(G) = 3S_6(G')$.

We may now conclude by induction on $|G|$ that S_6 is arithmetic, satisfying $\mathcal{N}(S_6(G)) = 3^m$ for $|G| + 1 \leq m \leq 2|G|$. For there is a single graph G on one vertex (disregarding loops), and $\mathcal{N}(S_6(G)) = 3^2$ in this case. If $|G| = 2$ then there could be 0, 1, or 2 edges, giving $\mathcal{N}(S_6(G)) = 3^4$, 3^3, or 3^3. Finally if $|G| > 2$ and G has edges, then we may apply one of the above operations and invoke the induction hypothesis, while if G has no edges then $\mathcal{N}(S_6(G)) = 3^{2|G|}$.

The argument showing that S_7 is arithmetic proceeds along the same lines, so we will utilize the notation above, except that $d = \sum r_k$ will no longer be exactly the degree of u. If there is a vertex for which $d \not\equiv 0 \bmod 3$, then we sum over its color $q = 0$, 1, 2. Recalling that $f_7(x, y) = x^2 + xy + y^2 + x + y \equiv (x - y)^2 + x + y$, we find that the resulting exponent on ω within the sum is

$$2lq + \sum_{k=1}^{n} r_k(q^2 + qa_k + a_k^2 + q + a_k) = dq^2 + \left(2l + d + \sum_{k=1}^{n} r_k a_k\right) q + \sum_{k=1}^{n} r_k(a_k^2 + a_k). \tag{25}$$

Substituting $q \mapsto q + d(2l + d + \sum r_k a_k)$ eliminates the linear term mod 3. A bit of algebra reveals that the expression we obtain may be rewritten as

$$dq^2 + 2d(2l + d)^2 + \sum_{j<k} dr_j r_k \left((a_j - a_k)^2 + a_j + a_k\right) + \sum_{k=1}^{n} 2a_k \cdot r_k(dl - dr_k + 2). \tag{26}$$

Therefore if we modify G by deleting vertex u along with all edges and loops at u, include $dr_j r_k$ additional edges between v_j and v_k, and include $r_k(dl - dr_k + 2)$ more loops at v_k, then the resulting graph G' satisfies

$$S_7(G) = \omega^{2d(2l+d)^2}(1 + 2\omega^d) S_7(G'). \tag{27}$$

It follows that $\mathcal{N}(S_7(G)) = 3\mathcal{N}(S_7(G'))$, as before.

Now suppose that $d \equiv 0$ for all vertices of G and there are vertices u_1 and u_2 with $r \not\equiv 0$ edges between them. Employing the same notation as before, the exponent of ω for a given coloring is

$$r((q_1 - q_2)^2 + q_1 + q_2) + 2l_1 q_1 + 2l_2 q_2 + \sum_{k=1}^{n} r_k((q_1 - a_k)^2 + q_1 + a_k) + s_k((q_2 - a_k)^2 + q_2 + a_k). \quad (28)$$

Since $r + \sum r_k \equiv r + \sum s_k \equiv 0$ the q_1^2 and q_2^2 terms vanish, along with many of the q_1 and q_2 terms. Since we are summing over the colors q_1 and q_2 we may substitute $q_1 \mapsto q_1 - r(\sum s_k a_k + 2l_2)$ and $q_2 \mapsto q_2 - r(\sum r_k a_k + 2l_1)$. The resulting expression can be rewritten as

$$r q_1 q_2 + 2r l_1 l_2 + \sum_{j<k} 2r(r_j s_k + r_k s_j)\left((a_j - a_k)^2 + a_j + a_k\right) + \sum_{k=1}^{n} 2a_k \cdot 2r(l_2 r_k + l_1 s_k + r_k s_k). \quad (29)$$

Hence if we create G' by deleting vertices u_1 and u_2 from G, along with all edges and loops at these vertices, and including $2r(r_j s_k + r_k s_j)$ more edges between v_j and v_k and $2r(l_2 r_k + l_1 s_k + r_k s_k)$ extra loops at v_k, then $S_7(G) = 3\omega^{2rl_2l_2} S_7(G')$. But $\mathcal{N}(S_7(G)) = 0$ or 3 when $|G| = 1$ and $\mathcal{N}(S_7(G)) = 0$, 27, or 81 when $|G| = 2$. Hence just as with graph sum S_6, we may now conclude by induction that S_7 is arithmetic, satisfying $\mathcal{N}(S_7(G)) = 0$ or 3^m for $|G| + 1 \leq m \leq 2|G|$.

In summary, we have now proven our main result:

Theorem 8 *Let S_M be a graph sum given by a symmetric 3×3 matrix M whose entries are cube roots of unity. Then S_M is arithmetic if and only if M is generated by a symmetric polynomial $f(x, y) = a(x^2 + y^2) + bxy + c(x + y) + d$ with integer coefficients, in the sense that the entries of M are $m_{jk} = \omega^{f(j,k)}$ for $0 \leq j, k \leq 2$.*

6 Further inquiry

Given that fully half of the preceding pages were devoted to the proof of seven separate cases, one would be justified in speculating as to whether there might be a more unified approach to proving that graph sums generated by symmetric polynomials are arithmetic. This author was unable to find such an approach (but not for lack of trying). However, a more versatile technique will be necessary for making further progress, such as proving the conjecture outlined below.

Of course, it is also natural to wonder what happens for level n graph sums when $n > 3$. Do the operations in Proposition 2 still produce precisely the classes of equivalent graph sums? How many nonequivalent level n graph sums are there? How should one define level n arithmetic graph sums, and to what extent can they be characterized using row products, determinants, or symmetric polynomials? Experience suggests that we should first answer answer these questions for primes, then prime powers, and finally for any value of n.

So let p be an odd prime, and let $\xi_p = e^{2\pi i/p}$ be a primitive p^{th} root of unity. Then level p graph sums will have values in $\mathbb{Z}[\xi_p]$, and there is a corresponding norm $\mathcal{N} : \mathbb{Z}[\xi_p] \to \mathbb{Z}$. It seems clear, in this case at least, to declare that a level p graph sum is arithmetic if and only if $S_M(G) = 0$ or p^k for some positive integer k. Computational evidence for $p = 5$ suggests that symmetric polynomials continue to characterize arithmetic graph sums. Therefore we propose

Conjecture 9 *Let p be an odd prime, and let S_M be the graph sum given by a symmetric $p \times p$ matrix M whose entries are p^{th} roots of unity. Then S_M is arithmetic if and only if M is generated by a symmetric polynomial $f(x, y) \in \mathbb{Z}/p\mathbb{Z}[x, y]$ having degree two or less, where the entries of M are $m_{jk} = \xi_p^{f(j,k)}$ for $0 \le j, k < p$. In particular, there are exactly p^4 such graph sums.*

It is less clear how to proceed if n is a power of a prime p, although presumably one would still require arithmetic graph sums to have norms that are a power of p. A fairly detailed examination of the case $n = 4$ reveals that there are either 1480, or 1072, or some other number of level four arithmetic graph sums, depending upon the criterion used for declaring a graph sum to be arithmetic. Regardless, there are perfectly acceptable arithmetic graph sums which do not seem to be given by symmetric polynomials. On a more tantalizing note, there are exactly $128 = 2^7$ level four arithmetic graph sums given by matrices all of whose entries are ± 1, and $32 = 2^5$ level *two* arithmetic graph sums whose matrices involve ± 1, $\pm i$; that is, *fourth* roots of unity. But this opens yet another can of worms, in which the size of the matrix no longer needs to match the root of unity employed.

On the subject of arbitrary graph sums, we mention only that given relatively prime positive integers n, n' and graph sums at these levels given by matrices M, M', there is a relatively straightforward way to construct an $nn' \times nn'$ matrix M'' giving a level nn' graph sum satisfying

$$S_{M''}(G) = S_M(G) \cdot S_{M'}(G) \tag{30}$$

for all multigraphs G. Perhaps this construction should be utilized to define level n arithmetic graph sums when n has two or more distinct prime factors. On the other hand, it might be better to settle on an analogous definition to the previous cases, then show that our construction gives all (or a subset of) the level nn' arithmetic graph sums.

Let us assume for the time being that we have arrived at the proper definition of an arithmetic graph sum. Then given a multigraph G, we have a countable set of values $S_M(G)$ where S_M ranges over all arithmetic graph sums. As the combinatorial interpretations above indicate, these values ought to provide a great deal of information about G, modulo every prime p. Ultimately, it would be desirable to organize these values in a coherent manner in the form of a single mathematical object associated to G that encodes all of this information. Ideally, this object would be a complete invariant for multigraphs that would provide a powerful technique for studying them. This program is likely a bit optimistic, as suggested by [1] or [6]. However, one can always hope.

Appendix

Level two graph sums are given by 2×2 symmetric matrices whose entries are ± 1. There are eight such matrices, representing three nonequivalent level two graph sums. These may be presented in terms of subgraphs, as done for the motivating example at the outset. Thus let G be a simple graph with loops. Recall that an induced subgraph $G' \subset G$ consists of a subset of the vertices of G, possibly empty, along with all edges of G both of whose endpoints are contained in this subset. We also define the complementary subgraph $\overline{G'}$ as the induced subgraph on all vertices not in G'. As before, let $e(G')$ denote the number of edges of G', and also define $e(G', \overline{G'})$ to be the number of edges with one vertex in each of G' and $\overline{G'}$. The reader may verify that we can then write the graph sums corresponding to the given matrices as a sum over subgraphs, as shown below.

$$\begin{array}{lll} S_1(G) = \displaystyle\sum_{G' \subset G} 1 & S_2(G) = \displaystyle\sum_{G' \subset G} (-1)^{e(G', \overline{G'})} & S_3(G) = \displaystyle\sum_{G' \subset G} (-1)^{e(G')} \\ M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & M_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & M_3 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{array} \tag{31}$$

Theorem 10 *All level two graph sums are arithmetic.*

Proof: Since there are $2^{|G|}$ induced subgraphs of G, clearly $S_1(G) = 2^{|G|}$ for any multigraph G, hence S_1 is arithmetic. We next claim that $S_2(G) = 0$ unless every vertex of G has even degree, in which case $S_2(G) = 2^{|G|}$ as well. For suppose that a vertex v has odd degree. Then given any pair of induced subgraphs G' and $\overline{G'}$, the parity of $e(G', \overline{G'})$ will be reversed by moving v from one of these subgraphs to the other. Hence the terms in the sum for $S_2(G)$ will cancel in pairs. On the other hand, if all vertices of G have even degree then it is easy to show that $e(G', \overline{G'})$ must also be even for all $G' \subset G$. This follows since

$$e(G', \overline{G'}) = \Big(\sum_{v \in V(G')} \deg(v) \Big) - 2e(G'),$$

and both quantities on the right are even. Thus S_2 is also arithmetic.

The demonstration that S_3 is arithmetic is somewhat less routine. Our strategy will be to successively isolate edges of G without altering the value of $S_3(G)$. Thus suppose that vertices u_1 and u_2 (possibly having loops) are joined by an edge. We construct a modified graph G' as follows. For each pair of edges $u_1 v$ and $u_2 w$ that occur in G, "add" an edge vw. (This adding is done mod 2, so that if edge vw already exists then it is erased.) It is possible that $v = w$, in which case a loop is added at v. Furthermore, it may be that the pair $u_1 w$ and $u_2 v$ also occur, in which case edge vw will be added twice. This process is illustrated in the first step of Fig. 4. Next "add" a loop at vertex v whenever edge $u_1 v$ exists and there is a loop at u_2, or vice-versa. Again, it is possible

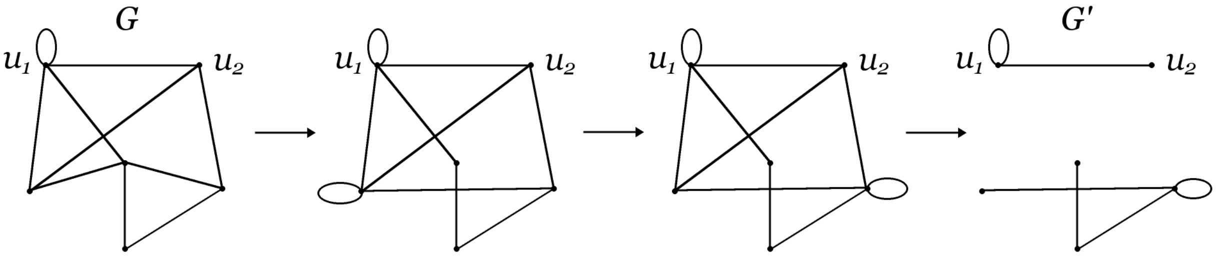


Figure 4: The process of isolating edge u_1u_2 without altering the value of $S_3(G)$.

for two loops to be introduced at v, if there are loops at both u_1 and u_2 and edges u_1v and u_2v both occur. Finally, delete all edges from u_1 and u_2 to the remaining vertices of G. This process is illustrated in the final steps of Fig. 4.

We claim that $S_3(G) = S_3(G')$. Suppose initially that there are no loops at u_1 or u_2, and let $H \subset G$ be a subgraph not containing u_1 or u_2. If we denote the number of edges from u_1 and u_2 to H by a and b, then the contribution to the sum $S_3(G)$ from the subgraphs H, $H \cup u_1$, $H \cup u_2$, and $H \cup u_1 \cup u_2$ will be

$$(-1)^{e(H)}\Big(1 + (-1)^a + (-1)^b + (-1)^{a+b+1}\Big). \tag{32}$$

On the other hand, within G' the subgraph on the vertex set of H has gained a total of ab additional edges. Hence the corresponding terms in the sum for $S_3(G')$ will be

$$(-1)^{e(H)+ab}\Big(1 + 1 + 1 + (-1)\Big) = 2(-1)^{ab}(-1)^{e(H)}. \tag{33}$$

If either of a or b is even then the above expressions both reduce to $2(-1)^{e(H)}$, while if both a and b are odd then the expressions equal $-2(-1)^{e(H)}$. It follows that we may split the sums for $S_3(G)$ and $S_3(G')$ into sets of four terms, one set for each such subgraph H, so that corresponding sets of terms have the same sum. Therefore $S_3(G) = S_3(G')$.

Next suppose that there is a single loop, say at u_1. In this case we must verify that

$$(-1)^{e(H)}\Big(1 + (-1)^{a+1} + (-1)^b + (-1)^{a+b+2}\Big) = (-1)^{e(H)+ab+b}\Big(1 + (-1) + 1 + 1\Big).$$

(Recall that b loops are added to the vertices of H due to the loop at u_1.) But the two sides of this equality may be obtained by replacing a by $a+1$ in (32) and (33), thus establishing the identity. Finally, if there are loops at both u_1 and u_2 then we must show that

$$(-1)^{e(H)}\Big(1 + (-1)^{a+1} + (-1)^{b+1} + (-1)^{a+b+3}\Big) = (-1)^{e(H)+ab+a+b}\Big(1 + (-1) + (-1) + (-1)\Big),$$

which follows by replacing a and b by $a+1$ and $b+1$ in (32) and (33).

By successively isolating edges we eventually reach a graph G' all of whose connected components consist of a vertex, a vertex with loop, or an edge with 0, 1, or 2 loops. The value of S_3 on these components is 2, 0, 2, 2, and -2, respectively. Hence $S_3(G') = 0$ or $\pm 2^m$ for a positive integer m. Furthermore, when the value is nonzero, each vertex or pair of vertices contributes a factor of 2 to the value, so we deduce that $\left\lfloor \frac{|G|+1}{2} \right\rfloor \leq m \leq |G|$. $\square$

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