A formula for the Mahler measure of $axy + bx + cy + d$

S. Vandervelde
Department of Mathematics, University of Chicago, Chicago, IL, 60637

Abstract

We introduce a formula for the Mahler measure of $axy + bx + cy + d$ with complex coefficients a, b, c, and d and give examples which demonstrate a connection with L-functions. We then prove a generalization of Maillot's formula when the coefficients are real. Next we discuss operations on the coefficients which fix the Mahler measure. Finally, we prove an alternate formulation of the main result in order to calculate the Mahler measure of a two-parameter family of polynomials in three variables.

keywords: Mahler measure, Bloch-Wigner dilogarithm, L-functions, Jensen's formula, polylogarithm, cyclic quadrilateral

1 Introduction

The derivation of explicit expressions for the Mahler measure $m(P)$ of Laurent polynomials in two variables has been the focus of some mathematical attention lately. Boyd [Bo1] and Rodriguez Villegas [RV] have produced polynomials for which $m(P)$ may be expressed in terms of the L-function of a quadratic character or of an associated elliptic curve. Smyth [Sm] has also developed an explicit formula for the Mahler measure of a family of polynomials in three variables with real parameters. Furthermore, Boyd [Bo2] has made the connection between Mahler measure and volumes of hyperbolic manifolds. In [Ma], Maillot develops an elegant formula for $m(ax + by + c)$ which involves the angles of the triangle whose sides have lengths $|a|$, $|b|$, and $|c|$. Motivated by the fact that linear fractional transformations also possess a geometrical flavor, we sought an analogous formula for $m(axy + bx + cy + d)$. In this paper we present a general formula, use it to prove a direct generalization of Maillot's result, and also discuss some of its other implications.

The Mahler measure $m(P)$ of a Laurent polynomial $P(x, y) \in \mathbb{C}[x, x^{-1}, y, y^{-1}]$ calculates the average value of $\log |P(x, y)|$ over $\mathbb{T}^2$, the subset of $\mathbb{C}^2$ consisting of all pairs (x, y) with $|x| = |y| = 1$. In other words,

$$m(P) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |P(x, y)| \, \frac{dx}{x} \, \frac{dy}{y} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |P(e^{is}, e^{it})| \, ds \, dt.$$

The Mahler measure of a Laurent polynomial in n variables is defined similarly. The one-variable case is essentially completely understood, as Jensen's formula may be employed to show that for $P(x) = a(x - \alpha_1) \cdots (x - \alpha_d)$,

$$m(P) = \log |a| + \sum_{k=1}^{d} \log^+ |\alpha_k|,$$

where as usual $\log^+ x = 0$ for $x < 1$ while $\log^+ x = \log x$ when $x \geq 1$. In particular, we have $m(ax + b) = \max(\log |a|, \log |b|)$ for $a, b \in \mathbb{C}^*$.

Recall that for $n \geq 2$ the general polylogarithm function $\mathrm{L}i_n(z)$ is given by

$$\mathrm{L}i_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \qquad |z| \leq 1.$$

The dilogarithm $\mathrm{L}i_2(z)$ may be extended to an analytic function on $\mathbb{C}$ except for a branch cut along the real axis from 1 to ∞. A modification of this function leads to the Bloch-Wigner dilogarithm, a real analytic function that can be extended to all of $\mathbb{P}^1(\mathbb{C})$, vanishing on the real axis and at infinity. It is defined as

$$D(z) = \mathrm{I}m(\mathrm{L}i_2(z)) + \log|z|\arg(1-z), \qquad z \in \mathbb{C} \setminus [1,\infty),$$

This function satisfies a number of identities such as $D(\bar{z}) = D(1/z) = -D(z)$ and the five-term relation

$$D(z) + D(w) + D\left(\frac{1-w}{z}\right) + D\left(\frac{z+w-1}{zw}\right) + D\left(\frac{1-z}{w}\right) = 0 \tag{1}$$

for any $z, w \in \mathbb{C}^*$. (See [Bl] or [Za] for a nice introduction to this function.) The Bloch-Wigner dilogarithm appears as the primitive of a certain differential 1-form, which enables us to perform the integration in the proof of our formula. The resulting expression for Mahler measure may be expressed in terms of either of these dilogarithms; each formulation has its advantages.

2 Statement of the formula

Although the expression $m(axy + bx + cy + d)$ is "lexicographically correct," our notation will be simplified if we instead compute $m(cxy - dy - ax + b)$. That these two quantities are equal follows from the fact that multiplying by x^{-1} and then making the change of variables $(x, y) \mapsto (-y^{-1}, -x)$ transforms the former polynomial into the latter, but does not affect the value of the integral used to define Mahler measure. Therefore we will only work with $P(x, y) = cxy - dy - ax + b$. Also, in the interest of handling special cases separately, we will initially suppose that the coefficients are non-zero and that $ad - bc \neq 0$, then later extend our formula to include all values of the coefficients for which $m(P)$ is defined.

Solving $P(x, y) = 0$ for y we find $y = (ax - b)/(cx - d)$, which is a linear fractional transformation since $ad - bc \neq 0$. In what follows, we will always treat y as this particular function of x, except when y is used as a dummy variable of integration. The image of the unit circle $|x| = 1$ under the transformation $x \mapsto y$ will be a circle (or line) which we will denote by $\mathcal{T}$. The position of $\mathcal{T}$ relative to the unit circle $|y| = 1$ distinguishes between two possibilities. The "intersection" case occurs when $\mathcal{T}$ intersects both the interior and exterior of the unit circle, while the "non-intersection" case encompasses all other positions of $\mathcal{T}$. This distinction may be phrased more algebraically by noting that there will be either zero, one, two, or infinitely many pairs $(x, y) \in \mathbb{C}^2$ satisfying $P(x, y) = 0$ and $|x| = |y| = 1$. The intersection case occurs if there are exactly two such pairs, otherwise we have the non-intersection case.

In the intersection case denote the two pairs by (x_1, y_1) and (x_2, y_2). Those values of $x \in \mathbb{T}^1$ for which $|y| \geq 1$ constitute a path τ on the unit circle with the usual counterclockwise orientation. Likewise those x for which $|y| \leq 1$ form a complementary arc τ'. Order the pairs so that τ has initial point x_1 and terminal point x_2, and vice-versa for τ'. To each of the coefficients we now associate a point and an angle. Set $z_a = b/a$, $z_b = \bar{a}/\bar{b}$, $z_c = d/c$, and $z_d = \bar{c}/\bar{d}$. Let α be the "winding angle" of the path τ with respect to z_a, which may be defined formally as

$$\mathrm{W}ind(\tau, z_a) = \mathrm{Im}\int_\tau \frac{dz}{z - z_a}, \qquad z_a \notin \tau. \tag{2}$$

Equivalently, let α be the directed angle $x_1 z_a x_2$ with value in $(0, 2\pi)$ when z_a is inside the unit circle, and with value in $(-\pi, \pi)$ for z_a outside the unit circle. We define β in the same way, while for γ we use the directed angle $x_2 z_c x_1$ instead, and similarly for δ. These correspond to the winding angles of τ' with respect to z_c and z_d. Note that $y = 0$ when $x = z_a$, so z_a

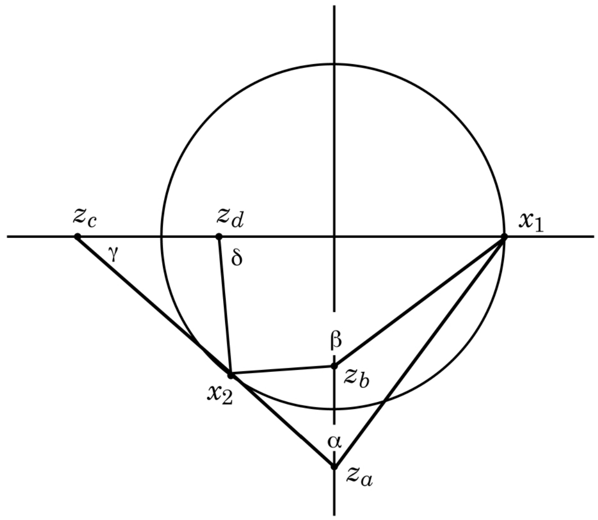

Figure 1: Location of points and angles for $m(2xy + 3y - 3ix + 4)$ computation.

cannot lie on τ. Also, if z_a is a point on the unit circle then $z_a = z_b$, so $z_b \notin \tau$ either. Similar considerations apply to z_c and z_d, so the winding angle is always well-defined.

We are now in a position to state a formula for $m(P)$.

Theorem 1 *Let $P(x, y) = cxy - dy - ax + b$ for a, b, c, $d \in \mathbb{C}^*$ such that $ad - bc \neq 0$. With the notation introduced above we have*

$$m(P) = \max(\log|a|, \log|b|, \log|c|, \log|d|) \tag{3}$$

in the non-intersection case, while

$$\begin{aligned} 2\pi m(P) &= D\left(\frac{cx_2}{d}\right) - D\left(\frac{cx_1}{d}\right) - D\left(\frac{ax_2}{b}\right) + D\left(\frac{ax_1}{b}\right) \\ &\quad + \alpha\log|a| + \beta\log|b| + \gamma\log|c| + \delta\log|d| \end{aligned} \tag{4}$$

in the intersection case.

We remark that this expression possesses the properties one would expect of a formula for Mahler measure. For instance, we shall see that $\alpha + \beta + \gamma + \delta = 2\pi$, so that for $\lambda \in \mathbb{C}^*$, our formula predicts that $m(\lambda P) = \log|\lambda| + m(P)$, as it should according to the definition.

3 Two examples

In order to illustrate the constructions involved in the theorem we will evaluate $m(P)$ for $P(x, y) = 2xy + 3y - 3ix + 4$ in terms of familiar functions. In this case we have $a = 3i$, $b = 4$, $c = 2$, and $d = -3$. To find (x, y) satisfying $P(x, y) = 0$ and $|x| = |y| = 1$ we equate $y\bar{y} = 1$, where $y = (3ix - 4)/(2x + 3)$, then use $\bar{x} = x^{-1}$. The resulting quadratic equation has two roots, $x_1 = 1$ and $x_2 = -\frac{1}{5}(3 + 4i)$. These are given in the correct order since $|y| = 5$ when $x = -1$, so $-1 \in \tau$. We now plot $z_a = -\frac{4}{3}i$, $z_b = -\frac{3}{4}i$, $z_c = -\frac{3}{2}$, and $z_d = -\frac{2}{3}$, as shown in Fig. 1. One finds that $\alpha = \delta = \arctan(12)$, $\beta = \pi - \arctan(\frac{32}{51})$, and $\gamma = \arctan(\frac{8}{9})$. Thus we conclude that

$$\begin{aligned}2\pi m(P) &= D(\tfrac{6+8i}{15}) - D(-\tfrac{2}{3}) - D(\tfrac{12-9i}{20}) + D(\tfrac{3i}{4}) + \\ &\quad + (2\pi - 2\arctan(\tfrac{32}{51}) + \arctan(\tfrac{8}{9}))\log 2 + 2\arctan(12)\log 3.\end{aligned}$$

These two quantities do in fact agree numerically.

Next we will show that $m(ixy + x + y + 1) = \frac{1}{4}L'(\chi, -1)$, where $\chi = \left(\frac{-2}{p}\right)$ is the quadratic Dirichlet character associated with $\mathbb{Q}(\sqrt{-2})$, namely

$$\chi(n) = \begin{cases} 1 & n \equiv 1, 3 \bmod 8 \\ -1 & n \equiv -1, -3 \bmod 8 \\ 0 & n \ even \end{cases}.$$

Using the same technique as before we find that $x_1 = \xi^{-1}$ and $x_2 = \xi^3$, where $\xi = e^{\pi i/4}$. The values of the angles are irrelevant since the coefficients each have absolute value one. Thus for $P(x, y) = ixy + x + y + 1$ we find

$$\begin{aligned}2\pi m(P) &= D(-i\xi^3) - D(-i\xi^{-1}) - D(-\xi^3) + D(-\xi^{-1}) \\ &= 2D(\xi) + 2D(\xi^3) \\ &= 2\ \mathrm{Im}\sum_{k=1}^{\infty}\frac{1}{k^2}(\xi^k + \xi^{3k}) \\ &= 2\sqrt{2}\ L(\chi, 2).\end{aligned}$$

Recall that the L-series of a character χ is defined by

$$L(\chi, s) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \qquad \mathrm{Re}(s) > 1$$

and can be extended to a meromorphic function on $\mathbb{C}$. As shown in [La], if χ is a primitive Dirichlet character with conductor N and Gauss sum $W(\chi)$ then $L(\chi, s)$ satisfies the functional equation

$$L(\bar{\chi}, 1-s) = L(\chi, s)\left(\frac{N}{2\pi}\right)^s \Gamma(s) W(\chi)^{-1}\left(e^{\frac{\pi i s}{2}} + \chi(-1)e^{\frac{-\pi i s}{2}}\right). \tag{5}$$

In our case $\chi(-1) = -1$, so the final factor vanishes at $s = 2$. Taking derivatives and evaluating at $s = 2$ yields $L'(\chi, -1) = \frac{4\sqrt{2}}{\pi}L(\chi, 2)$ since $N = 8$ and $W(\chi) = 2i\sqrt{2}$. Combining these computations yields

$$m(ixy + x + y + 1) = \frac{1}{4}L'(\chi, -1). \tag{6}$$

4 Proof of the formula

We first address the intersection case. So that we may apply Jensen's formula we write $P(x, y) = (cx - d)\left(y - \dfrac{ax - b}{cx - d}\right)$, yielding

$$\begin{aligned}m(P) &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\left(\log|cx - d| + \log\left|y - \frac{ax - b}{cx - d}\right|\right)\frac{dy}{y}\frac{dx}{x} \\ &= m(cx - d) + \frac{1}{2\pi i}\int_{\mathbb{T}^1}\log^+\left|\frac{ax - b}{cx - d}\right|\frac{dx}{x}.\end{aligned} \tag{7}$$

Let $\mathcal{C}$ be the curve defined by $P(x,y)=0$, and note that $\mathcal{C}\cong\mathbb{P}^1(\mathbb{C})$ via the coordinate function x. We will base our computations in this projective plane. Following [BRV] we introduce a closed, real-valued differential 1-form $\eta(f,g)$ on $\mathcal{C}$ for each ordered pair of rational functions $f,g\in\mathbb{C}(\mathcal{C})^*$. It is given by

$$\eta(f,g)=\mathrm{Im}\left(\log|f|\ \frac{dg}{g}-\log|g|\ \frac{df}{f}\right), \tag{8}$$

defined at all points of $\mathcal{C}$ where neither f nor g has a zero or pole. As may readily be verified, η is bimultiplicative and skew-symmetric in its arguments. We will need the fact that $\eta(f,1-f)$ is exact with primitive $D\circ f$.

Since the integrand of (7) is zero on the portion of $\mathbb{T}^1$ for which $|y|<1$ we may write

$$\frac{1}{2\pi i}\int_{\mathbb{T}^1}\log^+\left|\frac{ax-b}{cx-d}\right|\ \frac{dx}{x}=\frac{1}{2\pi}\ \mathrm{Im}\int_\tau\log|y|\ \frac{dx}{x}=-\frac{1}{2\pi}\int_\tau\eta(x,y),$$

where τ is the arc of the unit circle on which $|y|\geq 1$. Note that $\log|x|\ \frac{dy}{y}$ vanishes on τ since $|x|=1$. Using properties of η we have the identity

$$\begin{aligned}\eta(x,y)\quad=\quad&\eta\left(\frac{ax}{b},1-\frac{ax}{b}\right)-\eta\left(\frac{cx}{d},1-\frac{cx}{d}\right)\\&+\eta\left(x,\frac{b}{d}\right)-\eta\left(x-\frac{b}{a},\frac{b}{a}\right)+\eta\left(x-\frac{d}{c},\frac{d}{c}\right),\end{aligned}$$

so we may evaluate $\int_\tau\eta(x,y)$ in pieces. The first two terms have the primitive $D(ax/b)-D(cx/d)$, so their integral over τ is

$$D\left(\frac{ax_2}{b}\right)-D\left(\frac{ax_1}{b}\right)-D\left(\frac{cx_2}{d}\right)+D\left(\frac{cx_1}{d}\right)$$

by Stokes theorem. The other terms may be integrated directly; by definition

$$\int_\tau\eta\left(x,\frac{b}{d}\right)=\mathrm{Im}\int_\tau-\log\left|\frac{b}{d}\right|\ \frac{dx}{x}=-\theta\log\left|\frac{b}{d}\right|,$$

where θ is the winding angle of τ about the origin, i.e. the measure of the arc τ. In the same fashion we find that

$$\int_\tau\eta\left(x-\frac{b}{a},\frac{b}{a}\right)=-\alpha\log\left|\frac{b}{a}\right|,\quad\text{and}\quad\int_\tau\eta\left(x-\frac{d}{c},\frac{d}{c}\right)=-\gamma'\log\left|\frac{d}{c}\right|,$$

where α and γ' are the winding angles of τ with respect to $z_a=\frac{b}{a}$ and $z_c=\frac{d}{c}$. Note that unlike α, γ' is not the same as the angle γ defined in the statement of the theorem.

Collecting all these results and recalling that $m(P)=m(cx-d)-\frac{1}{2\pi}\int_\tau\eta(x,y)$, we find

$$\begin{aligned}2\pi m(P)\quad=\quad&D\left(\frac{cx_2}{d}\right)-D\left(\frac{cx_1}{d}\right)-D\left(\frac{ax_2}{b}\right)+D\left(\frac{ax_1}{b}\right)\\&+\alpha\log|a|+(\theta-\alpha)\log|b|-\gamma'\log|c|+(\theta-\gamma')\log|d|\\&+2\pi m(cx-d).\end{aligned} \tag{9}$$

In order to complete the proof we must relate the angles α, β, γ, and δ to θ.

Lemma 2 *Let τ be an arc of the unit circle, parameterized by $t \mapsto e^{it}$ for $t \in [\theta_1, \theta_2]$ where $\theta_1 < \theta_2$ and $\theta = \theta_2 - \theta_1 < 2\pi$. Given any $w \in \mathbb{C}^*$, let ϕ, ϕ' be the winding angles of τ with respect to w and $1/\bar{w}$. Then $\phi + \phi' = \theta$.*

Proof: According to (2) the winding angle ϕ' is given by

$$\phi' = \mathrm{Im} \int_\tau \frac{dz}{z - \frac{1}{\bar{w}}} = \mathrm{Im} \int_{\theta_1}^{\theta_2} \frac{ie^{it}\,dt}{e^{it} - \frac{1}{\bar{w}}}.$$

In order to find a more useful expression for ϕ' we observe that reflecting the entire picture over the real axis will negate the winding angle. The new path $\bar{\tau}$ will be parameterized by $t \mapsto e^{-it}$ with $t \in [\theta_1, \theta_2]$ and wind about $\frac{1}{w}$, so

$$\phi' = -\mathrm{Im} \int_{\theta_1}^{\theta_2} \frac{-ie^{-it}}{e^{-it} - \frac{1}{w}}\,dt = \mathrm{Im} \int_{\theta_1}^{\theta_2} \frac{i}{1 - e^{it}\frac{1}{w}}\,dt.$$

Adding the expression for ϕ yields

$$\phi + \phi' = \mathrm{Im} \int_{\theta_1}^{\theta_2} \frac{i}{1 - e^{it}\frac{1}{w}} + \frac{i}{1 - e^{-it}w}\,dt = \mathrm{Im} \int_{\theta_1}^{\theta_2} i\,dt = \theta,$$

as desired. Lest the geometry be completely obscured, we mention that the proof may also be accomplished by using two pairs of similar triangles and some casework. However, the above approach is more efficient.

It follows from the lemma that $\theta - \alpha = \beta$, giving the desired $\beta \log|b|$ term in (9). To handle γ' we consider two cases. If $|c| \geq |d|$ then z_c lies within the unit circle and $\gamma' = 2\pi - \gamma$. Furthermore, $m(cx - d) = \log|c|$ in this case. Therefore the final three terms of (9) reduce to

$$-(2\pi - \gamma)\log|c| + (2\pi - \gamma - \theta)\log|d| + 2\pi \log|c| = \gamma \log|c| + \delta \log|d|,$$

since $\gamma + \delta = 2\pi - \theta$, by the lemma. If instead $|c| \leq |d|$ then z_c lies outside the unit circle, so $\gamma' = -\gamma$, $m(cx - d) = \log|d|$, and the outcome is the same regardless, thereby proving our formula for $m(P)$ in the intersection case.

The non-intersection case requires considerably less analysis. We will need the following result, which essentially says that if $\mathcal{T}$ lies completely inside or on the unit circle $|y| = 1$, then a coefficient with maximal magnitude must appear in the denominator of $y = (ax - b)/(cx - d)$.

Lemma 3 *Let a, b, c, and d be complex numbers such that $d \neq 0$, and let $y = (ax - b)/(cx - d)$. If $|y| \leq 1$ for all $|x| = 1$ then* $\max(|a|, |b|) \leq \max(|c|, |d|)$.

Proof: Choose $\xi \in \mathbb{T}^1$ so that $c\xi/d$ is a positive real number. This ensures that $|c\xi + d| = |c| + |d|$ and $|c\xi - d| = \pm(|c| - |d|)$. By hypothesis $|y| \leq 1$ for $x = \pm\xi$, meaning that $|a\xi + b| \leq |c\xi + d|$ and $|a\xi - b| \leq |c\xi - d|$. Now observe that

$$2|a| = |(a\xi + b) + (a\xi - b)| \leq |(a\xi + b)| + |(a\xi - b)| \leq |c\xi + d| + |c\xi - d|,$$

and the latter equals either $2|c|$ or $2|d|$, so $|a| \leq \max(|c|, |d|)$. By considering $|(-a\xi + b) + (a\xi + b)|$ instead, we deduce that $|b| \leq \max(|c|, |d|)$ as well.

As shown in (7),

$$m(P) = m(cx - d) + \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ \left| \frac{ax - b}{cx - d} \right| \frac{dx}{x}.$$

If $\mathcal{T}$ lies completely inside or on the unit circle, then $\log^+|y| = 0$ for all $x \in \mathbb{T}^1$, so the above integral reduces to

$$m(P) = m(cx - d) = \max(\log|c|, \log|d|).$$

By the lemma we may now conclude that

$$m(P) = \max(\log|a|, \log|b|, \log|c|, \log|d|).$$

Conversely, if $\mathcal{T}$ lies on or outside the unit circle, then $|(cx-d)/(ax-b)| \leq 1$ for all $x \in \mathbb{T}^1$. The lemma now asserts that $\max(|c|, |d|) \leq \max(|a|, |b|)$. In this case $\log^+|y| = \log|y|$, so (7) reduces to

$$\begin{aligned} m(P) &= m(cx-d) + \frac{1}{2\pi i}\int_{\mathbb{T}^1} (\log|ax-b| - \log|cx-d|)\,\frac{dx}{x} \\ &= m(cx-d) + m(ax-b) - m(cx-d) \\ &= \max(\log|a|, \log|b|) \\ &= \max(\log|a|, \log|b|, \log|c|, \log|d|) \end{aligned}$$

as before. This establishes the formula in the non-intersection case, completing the proof of the theorem.

5 Degenerate cases

We first relax the requirement that $ad - bc \neq 0$. If we have $ad = bc$ then the image $\mathcal{T}$ of $\mathbb{T}^1$ under $x \mapsto (ax-b)/(cx-d)$ is the single point $y = a/c$. According to our original geometric criterion, we are in the non-intersection case. Lemma 3 still applies, and we conclude that

$$m(P) = \max(\log|a|, \log|b|, \log|c|, \log|d|).$$

To apply our result when one or more of the coefficients are zero we will use the fact that $m(P)$ is a continuous function of the parameters a, b, c, and d (see [Bo1]). Hence if $a = 0$, for example, we need only analyze our formulae as $a \to 0$. In the non-intersection case, it is clear that

$$\lim_{a\to 0} m(P) = \max(\log|b|, \log|c|, \log|d|).$$

In the intersection case, since $D(0) = 0$, the dilogarithm terms involving a will vanish. Furthermore, $z_a = b/a$ will go to infinity, so $\alpha \to 0$. More precisely, α subtends an arc of the unit circle, so $|\alpha| \leq 2\arcsin(|\frac{a}{b}|)$. But as $a \to 0$ we know that $\arcsin(|\frac{a}{b}|) \sim |\frac{a}{b}|$ and $|a|\log|a| \to 0$, so we deduce that $\alpha \log|a| \to 0$. Thus

$$\lim_{a\to 0} 2\pi m(P) = D\left(\frac{cx_2}{d}\right) - D\left(\frac{cx_1}{d}\right) + \beta\log|b| + \gamma\log|c| + \delta\log|d|.$$

In summary, our formulae will still hold when $a = 0$ if we agree to ignore any term involving a. Recall that $D(z)$ vanishes at infinity as well, so the same convention about ignoring terms will apply when any of b, c, or d are zero. Finally, note that difficulties such as interpreting $D(\frac{0}{0})$ will not arise, since if $a = b = 0$ or $c = d = 0$ then $ad - bc = 0$, giving the non-intersection case.

6 Real coefficients

When a, b, c, and d are real there is a single cyclic quadrilateral with sides of length $|a|$, $|b|$, $|c|$, and $|d|$ which encapsulates all the angles, lengths, and ratios involved in the expression for $m(cxy - dy - ax + b)$. This geometric formulation is the natural generalization of Maillot's elegant result for $m(ax + by + c)$ in [Ma]. However, in our case the Mahler measure does not depend solely on the norms of the coefficients, hence our restriction to real values.

We first explain how a value of the Bloch-Wigner dilogarithm may be associated to an oriented triangle. Let z_1, z_2, and z_3 be distinct points in the complex plane. We define the "dilogarithm of triangle $z_1z_2z_3$" as

$$D(z_1z_2z_3) = D\left(\frac{z_1 - z_2}{z_3 - z_2}\right).$$

Although the notation $D(z_1z_2z_3)$ could also be construed as the dilogarithm of the product of z_1, z_2, and z_3, it will be clear from the context which meaning is intended. The identities $D(z) = D(\frac{1}{1-z}) = D(\frac{z-1}{z})$ imply that $D(z_2z_3z_1)$ and $D(z_3z_1z_2)$ also share the same value. Furthermore, it is clear that the ratio above depends only on the shape of the triangle, not its size. Thus D can be viewed as a function on equivalence classes of similar triangles under orientation-preserving similarity transformations. Furthermore, $D(z_3z_2z_1) = -D(z_1z_2z_3)$ since $D(\frac{1}{z}) = -D(z)$, so reversing the orientation of a triangle negates its dilogarithm. Note that if the vertices are oriented positively (in counterclockwise order) then $(z_1 - z_2)/(z_3 - z_2)$ will lie in the upper half plane, so $D(z_1z_2z_3)$ will be positive. Lastly, $D(z_1z_2z_3) = 0$ if and only if the triangle is degenerate, i.e. z_1, z_2, and z_3 are collinear.

With this interpretation of the dilogarithm we can state Maillot's result in the following form:

Theorem 4 (Maillot) *Given $a, b, c \in \mathbb{C}$, suppose that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a non-degenerate triangle. Label the triangle ABC, oriented positively, with angles α', β', and γ' opposite the sides of length $|a|$, $|b|$, and $|c|$, respectively. In this case*

$$\pi m(ax + by + c) = D(ABC) + \alpha' \log|a| + \beta' \log|b| + \gamma' \log|c|. \tag{10}$$

When $|a|$, $|b|$, and $|c|$ do not satisfy the triangle inequalities,

$$m(ax + by + c) = \max(\log|a|, \log|b|, \log|c|).$$

To extend this result to our situation, we will need to consider the dilogarithm of an oriented cyclic quadrilateral. If z_1, z_2, z_3, and z_4 are the vertices of a cyclic quadrilateral then we define

$$D(z_1z_2z_3z_4) = D(z_1z_2z_3) + D(z_1z_3z_4).$$

The next lemma shows that this value depends only on the order in which the vertices are visited, not on which vertex we choose to label z_1.

Lemma 5 *Let z_1, z_2, z_3, and z_4 lie on a single circle in the complex plane. Then we have*

$$D(z_1z_2z_3) - D(z_2z_3z_4) + D(z_3z_4z_1) - D(z_4z_1z_2) = 0. \tag{11}$$

Proof: Since the z_i lie on a circle their cross-ratio is a real number λ. If we set $w_1 = z_1 - z_2$, $w_2 = z_3 - z_4$, $w_3 = z_1 - z_4$, and $w_4 = z_3 - z_2$ then this fact may be expressed as $w_1w_2/w_3w_4 = \lambda$, while the assertion becomes

$$D\left(\frac{w_1}{w_4}\right) - D\left(\frac{w_4}{w_2}\right) + D\left(\frac{w_2}{w_3}\right) - D\left(\frac{w_3}{w_1}\right) = 0.$$

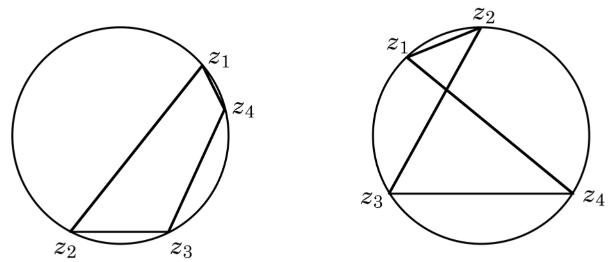

Figure 2: Positive orientation for cyclic quadrilaterals.

We claim that in the periodic sequence

$$1-\lambda, \quad \frac{w_2}{w_3}, \quad \frac{\frac{w_4}{w_2}}{\frac{w_4}{w_2}-1}, \quad \frac{\frac{w_3}{w_1}}{\frac{w_3}{w_1}-1}, \quad \frac{w_1}{w_4}, \quad 1-\lambda, \quad \frac{w_2}{w_3}, \quad \ldots$$

any three consecutive terms a_{j-1}, a_j, and a_{j+1} satisfy $a_{j-1}a_{j+1} = 1 - a_j$. This follows easily from the definition of λ and the observation that $w_1 + w_2 = w_3 + w_4$. Hence this sequence gives rise to a five term relation

$$D(1-\lambda) + D\left(\frac{w_2}{w_3}\right) + D\left(\frac{\frac{w_4}{w_2}}{\frac{w_4}{w_2}-1}\right) + D\left(\frac{\frac{w_3}{w_1}}{\frac{w_3}{w_1}-1}\right) + D\left(\frac{w_1}{w_4}\right) = 0.$$

The desired equation now follows from the identity $D(\frac{z}{z-1}) = -D(z)$ and the fact that D vanishes at real numbers.

As before, the dilogarithm remains constant under an orientation-preserving similarity transformation applied to the vertices, and reversing the orientation clearly negates the dilogarithm. Therefore we may use the sign of $D(z_1z_2z_3z_4)$ to define the positive orientation of a cyclic quadrilateral. (If the dilogarithm is zero then the quadrilateral is degenerate; this occurs if the vertices are collinear or, for the non-convex case, if opposite sides are congruent.) When $z_1z_2z_3z_4$ is convex this approach agrees with the usual convention that listing vertices in counterclockwise order gives the positive orientation. If $z_1z_2z_3z_4$ is cyclic and non-convex, and thus self-intersecting, there is still a simple geometric method for establishing the positive orientation. There will be a unique edge of minimal length; label it z_1z_2 so that the minor arc from z_1 to z_2 is traced out in a clockwise direction, then continue labeling the remaining vertices in order along the edges, as shown in Fig. 2.

We now associate an angle measure in $(-\frac{\pi}{2}, \pi)$ to each directed side of the quadrilateral, namely half the measure of the arc obtained by moving along the circle in a counterclockwise sense from the initial point to the terminal point of the side. Note that such an angle measure may exceed $\frac{\pi}{2}$, as for side z_1z_2 of the convex example in Fig. 2. The single exception occurs with the shortest side of a non-convex cyclic quadrilateral, in which we agree to trace out the circle in a clockwise direction and take the resulting angle to be negative. With these conventions the sum of the four angles is always π.

Finally, we remark that the dilogarithm of a positively oriented, convex, cyclic quadrilateral depends only on its side lengths. This follows from observing that $D(z_1z_2z_3z_4)$ is not affected by reflecting point z_2 across the perpendicular bisector of z_1z_3, and noting that such reflections produce all congruence classes of cyclic quadrilaterals with given side lengths. This argument is just as applicable in the non-convex case, implying the same conclusion.

The preceding discussion allows us to present the generalization to Maillot's formula in a unified manner.

Theorem 6 *Let $P(x,y) = axy + bx + cy + d$ for $a, b, c, d \in \mathbb{R}^*$. Suppose that there exists a non-degenerate convex (resp. non-convex) cyclic quadrilateral with sides of length $|a|$, $|b|$, $|c|$, and $|d|$ when $abcd < 0$ (resp. $abcd > 0$). Let $ABCD$ be such a quadrilateral, oriented positively, and let α', β', γ', and δ' be the angles associated to these sides as described above. Then*

$$\pi m(P) = D(ABCD) + \alpha' \log |a| + \beta' \log |b| + \gamma' \log |c| + \delta' \log |d|. \tag{12}$$

Otherwise, in the non-quadrilateral case we have

$$m(P) = \max(\log |a|, \log |b|, \log |c|, \log |d|).$$

Proof: The position of the points $z_a = b/a$ and $z_c = d/c$ relative to the origin gives rise naturally to the convex and non-convex cases. First suppose that these points lie on opposite sides of the origin, so that $abcd < 0$. We claim that the intersection case applies if and only if $|a|$, $|b|$, $|c|$, and $|d|$ are the lengths of the sides of a non-degenerate convex cyclic quadrilateral; that is, if and only if the greatest length is strictly less than the sum of the other three lengths. Since the coefficients are real the image circle $\mathcal{T}$ will be symmetric about the real axis, so we may check for intersection of $\mathcal{T}$ with $\mathbb{T}^1$ simply by examining $y = (ax - b)/(cx - d)$ at $x = \pm 1$. There will be no intersection precisely when $|y| \geq 1$ for $x \pm 1$, or when $|y| \leq 1$ for $x = \pm 1$.

To see that no intersection implies no quadrilateral, suppose that $|y| \leq 1$ for $x = \pm 1$, meaning $|a - b| \leq |c - d|$ and $|a + b| \leq |c + d|$. Without loss of generality, suppose $|c| \geq |d|$ and for sake of argument take $c > 0$. Then the two inequalities become $c \geq |a - b| + d$ and $c \geq |a + b| - d$. Either one or all three of a, b, and d are negative; considering the various possibilities shows that one of these two inequalities is equivalent to $|c| \geq |a| + |b| + |d|$, meaning no quadrilateral exists. The argument for $c < 0$ is similar, as is the case $|y| \geq 1$ if we focus on a instead of c. The converse is also straight-forward. If no quadrilateral exists, then an inequality such as $|a| \geq |b| + |c| + |d|$ holds. But this implies that

$$|c - d| \leq |c| + |d| \leq |a| - |b| \leq |a - b|,$$

so $|y| \geq 1$ when $x = 1$. In the same manner we find $|c + d| \leq |a + b|$, so $|y| \geq 1$ at $x = -1$ as well, which means we have the non-intersection case. The same reasoning applies when $|b|$, $|c|$, or $|d|$ is the largest. Therefore when z_a and z_c are on opposite sides of the origin, the intersection case is synonomous with the non-degenerate, convex, cyclic quadrilateral case.

In the intersection case a convex cyclic quadrilateral with sides of length $|a|$, $|b|$, $|c|$, and $|d|$ can be built using the points 0 (the origin), z_a, z_c, and x_2. First dilate triangle $z_a 0 x_2$ by a factor of $|a|$ to obtain a triangle with side lengths $|a|$, $|b|$, and $|ax_2 - b|$. Similarly, dilating triangle $z_c 0 x_2$ by a factor of $|c|$ yields a triangle with sides of length $|c|$, $|d|$, and $|cx_2 - d|$. But x_2 was defined by requiring that $|y| = 1$ when $x = x_2$, thus $|ax_2 - b| = |cx_2 - d|$. Hence we may rotate and translate the latter triangle until the sides of equal length coincide. Since z_a and z_c lie on either side of the origin, the resulting figure will be a convex quadrilateral which we label $ABCD$, in counterclockwise order. Because $\angle z_a 0 x_2$ and $\angle z_c 0 x_2$ are clearly supplementary, $ABCD$ is a cyclic quadrilateral. An example of this construction for $y = (3x - 1)/(2x + 5)$ is shown in Fig. 3.

When the coefficients of $P(x,y)$ are real, x_1 and x_2 will be conjugates, so (4) reduces to

$$\begin{aligned} \pi m(P) &= D\left(\frac{cx_2}{d}\right) - D\left(\frac{ax_2}{b}\right) + \\ &+ \frac{1}{2}(\alpha \log |a| + \beta \log |b| + \gamma \log |c| + \delta \log |d|). \end{aligned} \tag{13}$$

We claim that when $z_a > 0$ and $z_c < 0$, arc τ passes through $x = -1$ on the unit circle as in Fig. (3), placing x_2 in the lower half plane. Suppose otherwise; then $|y| < 1$ at $x = -1$,

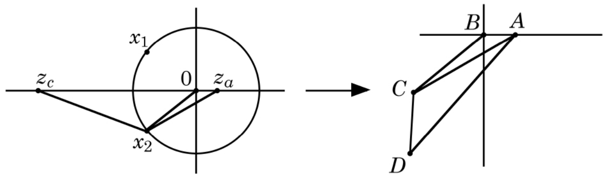

Figure 3: Construction of cyclic quadrilateral $ABCD$.

hence $|a+b| < |c+d|$. Since a and b have the same sign while c and d have opposite signs, we conclude that $|a|+|b| < \pm(|c|-|d|)$, leading to a non-quadrilateral inequality, a contradiction. In the same manner we find that if $z_a < 0$ and $z_c > 0$ then τ passes through $x = 1$ and x_2 is in the upper half plane. Either way, triangle $x_2 0 z_a$ is oriented negatively, while triangle $x_2 0 z_c$ is oriented positively. Therefore

$$D(ABCD) = D(x_2 0 z_c) - D(x_2 0 z_a) = D\left(\frac{cx_2}{d}\right) - D\left(\frac{ax_2}{b}\right),$$

matching the first two terms of (13).

We will next demonstrate that the angles occuring in (13) correspond to the appropriate arc lengths on the circumcircle of $ABCD$. By the foregoing discussion on the relative positions of z_a, z_c, x_1, and x_2, and recalling the definition of angles α, β, γ, and δ, we see that these angles must all be positive. Clearly $m\angle x_2 z_a 0 = \frac{1}{2}\alpha$ and $m\angle x_2 z_c 0 = \frac{1}{2}\gamma$. The similarity $\triangle x_2 0 z_z \sim \triangle z_b 0 x_2$ shows that $m\angle z_a x_2 0 = \frac{1}{2}\beta$. By the same reasoning we have $m\angle z_c x_2 0 = \frac{1}{2}\delta$. But these angles are inscribed in a circle, so we know that $m\angle x_2 z_a 0 = \alpha'$, and similarly for the remaining angles. Hence equations (12) and (13) agree.

The analysis of $m(P)$ when z_a and z_c lie on the same side of the origin follows the above reasoning closely, so we will be content to provide an outline of the argument. In this case $abcd > 0$, and the intersection case becomes synonomous with the non-degenerate, non-convex, cyclic quadrilateral case. We can construct a quadrilateral using triangles $x_2 0 z_a$ and $x_2 0 z_c$ in the same manner as before; due to the location of z_a and z_c it is guaranteed to be non-degenerate, self-intersecting, and cyclic. Triangle $x_2 0 z_c$ will be negatively oriented exactly when either $|c|$ or $|d|$ is the smallest side length, so labeling the quadrilateral in the positive sense will always result in

$$D(ABCD) = D(x_2 0 z_c) - D(x_2 0 z_a) = D\left(\frac{cx_2}{d}\right) - D\left(\frac{ax_2}{b}\right),$$

as before. Finally, elementary angle chasing reveals that the angles match in the manner desired. This completes the proof of the theorem.

7 Operations fixing $m(P)$

Returning now to the general setting with arbitrary complex coefficients, let us investigate operations on these coefficients which leave the Mahler measure unchanged. For example, since $m(\overline{P(x^{-1}, y^{-1})}) = m(P(x,y))$, we see that conjugating every coefficient fixes $m(P)$. It is also clear that the value of $m(P)$ is not affected by negating any two coefficients. Furthermore, permutations such as $(a,b,c,d) \mapsto (c,a,d,b)$ or $(a,b,c,d) \mapsto (b,a,d,c)$ do not affect the value of $m(P)$; this follows from the definition of Mahler measure and straight-forward changes of variables. However, there is another such operation which depends on the particular type of polynomial P we have been considering.

Proposition 7 *For $a, b, c, d \in \mathbb{C}$ not all zero, let $P(x, y) = cxy - dy - ax + b$. Then $m(P)$ is invariant under the map $(a, b, c, d) \mapsto (\bar{b}, \bar{a}, c, d)$.*

Proof: The assertion is clear in the non-intersection case, which we now take to include $ad - bc = 0$. For the intersection case, we first assume that all coefficients are non-zero. When $|x| = 1$ we have

$$|\bar{b}x - \bar{a}| = |b\bar{x} - a| = |ax - b|,$$

so the set of $x \in \mathbb{T}^1$ for which $|y| \geq 1$ does not change when we substitute $(a, b) \mapsto (\bar{b}, \bar{a})$ in $P(x, y)$. In particular, the values of x_1 and x_2 remain the same. Since z_a and z_b change places under this operation, the angles α and β are also interchanged. Of course, z_c, z_d, γ, and δ remain fixed. Therefore the contribution from the angle terms in (4) is constant. We need only verify that

$$-D\left(\frac{ax_2}{b}\right) + D\left(\frac{ax_1}{b}\right) = -D\left(\frac{\bar{b}x_2}{\bar{a}}\right) + D\left(\frac{\bar{b}x_1}{\bar{a}}\right). \tag{14}$$

But using properties of $D(z)$ we find that

$$D\left(\frac{\bar{b}x_2}{\bar{a}}\right) = -D\left(\frac{b\bar{x}_2}{a}\right) = D\left(\frac{a}{b\bar{x}_2}\right) = D\left(\frac{ax_2}{b}\right),$$

which implies (14). By continuity we may now conclude that $m(P)$ does not change under the given operation for any $a, b, c, d \in \mathbb{C}$ not all zero.

For a given 4-tuple (a, b, c, d) we refer to the positions occupied by a and d as the "outer spots," and the other two as the "inner spots." Let S be the set of all 4-tuples obtained from (a, b, c, d) by performing any combination of the following operations:

- an arbitrary permutation of the elements, provided we conjugate any element moving from an inner spot to an outer spot, or vice-versa,
- the negation of an even number of elements, or
- conjugation of all four elements.

Corollary 8 *For $a, b, c, d \in \mathbb{C}$ not all zero, let $P(x, y) = cxy - dy - ax + b$. Then $m(P)$ is invariant under the map $(a, b, c, d) \mapsto \mathbf{s}$ for any $\mathbf{s} \in S$.*

Proof: The set S consists of exactly those 4-tuples which may obtained from (a, b, c, d) by successively applying the operations discussed above.

We remark that some of these operations imply fairly complicated identities in light of (4). For example, the transposition $(a, c) \mapsto (\bar{c}, \bar{a})$ will preserve $m(P)$ according to the corollary. One can show that the angle terms in the formula still match up, although in a much less trivial manner than before. The corresponding equality among the dilogarithm terms is an eight-term relation in four variables, which in fact follows from a difference of two of the five-term relations mentioned in (1).

8 Extension to three variables

For this application we will need an alternate formulation of Theorem 1 more suitable for computations. When $|a| \leq |b|$ and $|c| \leq |d|$, so that points z_a and z_c are on or outside the unit circle, it is possible to obtain a simpler expression for $m(P)$. As we have just seen, this ordering is always possible, by replacing a and b by $\bar{b}$ and $\bar{a}$ if need be.

Corollary 9 *Let $P(x,y) = cxy - dy - ax + b$ for a, b, c, $d \in \mathbb{C}^*$ such that $ad - bc \neq 0$, $|a| \leq |b|$, and $|c| \leq |d|$. In the intersection case, let τ be the path on the unit circle $|x| = 1$ for which $|y| \geq 1$. Denote the endpoints of τ by x_1 and x_2 in counterclockwise order, and let θ be the measure of arc τ. Then*

$$\begin{aligned} 2\pi m(P) &= \operatorname{Im}\left[\operatorname{Li}_2\left(\frac{cx_2}{d}\right) - \operatorname{Li}_2\left(\frac{cx_1}{d}\right) - \operatorname{Li}_2\left(\frac{ax_2}{b}\right) + \operatorname{Li}_2\left(\frac{ax_1}{b}\right)\right] \\ &\quad +\theta \log|b| + (2\pi - \theta)\log|d|. \end{aligned} \tag{15}$$

Proof: Since $D(z) = \operatorname{Im}(\operatorname{Li}_2(z)) + \log|z| \arg(1-z)$, the origin of the terms involving Li_2 is clear. We must now account for the angles, i.e. show that

$$\log\left|\frac{a}{b}\right| \left(\arg\left(1 - \frac{ax_1}{b}\right) - \arg\left(1 - \frac{ax_2}{b}\right)\right) + \alpha \log|a| + \beta \log|b| \tag{16}$$

reduces to $\theta \log|b|$, and similarly for the terms involving c or d. Recall that α was defined as the winding angle of τ about z_a, which is equivalent to the measure of the directed angle $x_1 z_a x_2$ with value in $(-\pi, \pi)$ because z_a lies outside the unit circle. The transformation $z \mapsto 1 - z/z_a$ preserves directed angle measure, so we obtain the same angle using the points $1 - x_1/z_a$, 0, and $1 - x_2/z_a$ (in that order) instead. But the two non-zero points lie in the half-plane $\operatorname{Re}(z) > 0$ since $|z_a| \geq 1$ and $z_a \neq x_1$, $z_a \neq x_2$. Therefore

$$\alpha = \arg\left(1 - \frac{x_2}{z_a}\right) - \arg\left(1 - \frac{x_1}{z_a}\right)$$

gives the correct expression for α, with value in $(-\pi, \pi)$. It follows that (16) may be written as

$$-\alpha(\log|a| - \log|b|) + \alpha \log|a| + \beta \log|b|.$$

Since $\alpha + \beta = \theta$ by Lemma 2, we obtain $\theta \log|b|$ as desired. In the same manner, the terms involving c or d reduce to $(2\pi - \theta)\log|d|$.

This alternate formula will enable us to determine $m(P)$ for a family of four-term polynomials in three variables. We will initially write $P_z(x,y)$ rather than $P(x,y,z)$ to underscore our treatment of z as a parameter. To avoid unnecessary algebra, we will also assume that the coefficients are real, which will not result in any loss of generality.

Proposition 10 *Let $P(x,y,z) = rxy + y - rxz + 1$ with $0 \leq r \leq 1$. Then*

$$m(P) = \frac{2}{\pi^2}(\operatorname{Li}_3(r) - \operatorname{Li}_3(-r)). \tag{17}$$

Proof: Here $y = (rzx - 1)/(rx + 1)$, where we write $z = e^{it}$ for $t \in (-\pi, \pi)$. (We have neglected $z = -1$, which is permissible since we will be integrating with respect to z. In fact, $m(P_{-1}(x,y)) = 0$ by (3).) To find x for which $|x| = |y| = 1$ and $P_z(x,y) = 0$ we equate $y\bar{y} = 1$ and use $\bar{x} = x^{-1}$ as before to obtain a quadratic equation in x, whose roots are $x = \pm i|z+1|/(z+1)$. Because $t \in (-\pi, \pi)$ and $-1 \in \tau$ we have $x_1 = ie^{-it/2}$ and $x_2 = -ie^{-it/2}$. Note that $\theta = \pi$ since the roots are diametrically opposed for any z. Because $r \leq 1$, (15) applies, so we may write

$$\begin{aligned} 2\pi m(P_z(x,y)) &= \operatorname{Im}\Big[\operatorname{Li}_2(rie^{-it/2}) - \operatorname{Li}_2(-rie^{-it/2}) - \operatorname{Li}_2(-rie^{it/2}) + \\ &\qquad + \operatorname{Li}_2(rie^{it/2})\Big]. \end{aligned}$$

Since the arguments of $\mathrm{L}i_2$ are conjugates, this expression simplifies to

$$m(P_z(x,y)) = \frac{1}{\pi}\,\mathrm{Im}\left[\mathrm{L}i_2(rie^{it/2}) + \mathrm{L}i_2(rie^{-it/2})\right].$$

According to the definition of Mahler measure,

$$\begin{aligned} m(P(x,y,z)) &= \frac{1}{2\pi i}\int_{\mathbb{T}^1} m(P_z(x,y))\,\frac{dz}{z} \\ &= \frac{1}{2\pi^2}\,\mathrm{Im}\int_{-\pi}^{\pi} \mathrm{L}i_2(rie^{it/2}) + \mathrm{L}i_2(rie^{-it/2})\,dt. \end{aligned}$$

Since the series for $\mathrm{L}i_2(rie^{it/2})$ is absolutely convergent, we may integrate term by term to find that

$$\begin{aligned} \int_{-\pi}^{\pi} \mathrm{L}i_2(rie^{it/2})\,dt &= \sum_{k=1}^{\infty}\int_{-\pi}^{\pi}(ri)^k\,\frac{e^{ikt/2}}{k^2}\,dt \\ &= \left(\sum_{k=1}^{\infty}(ri)^k\,\frac{2e^{ikt/2}}{ik^3}\right)\Bigg|_{-\pi}^{\pi} \\ &= -2i\,\mathrm{L}i_3(rie^{it/2})\Big|_{-\pi}^{\pi} \\ &= 2i(\mathrm{L}i_3(r) - \mathrm{L}i_3(-r)). \end{aligned}$$

Integrating the second $\mathrm{L}i_2$ term yields the same result, from which (17) follows immediately. We note that the result may also be written

$$m(rxy + y - rxz + 1) = \frac{4}{\pi^2}\sum_{k\ \mathrm{odd}}\frac{r^k}{k^3}. \tag{18}$$

Using operations which preserve Mahler measure, such as those mentioned at the head of Section 2, one can transform $rxy + y - rxz + 1$ into $rx + ry + z + 1$. Observe that $m(\lambda P) = \log|\lambda| + m(P)$, and that substitutions such as $x \mapsto \xi x$ for $\xi \in \mathbb{T}^1$ do not affect Mahler measure. Therefore we are led to the following symmetrical formula, which we state without formal proof.

Corollary 11 *Given $a, b \in \mathbb{C}^*$ with $|a| \le |b|$, let $r = |a|/|b|$. Then*

$$m(ax + ay + bz + b) = \log|b| + \frac{2}{\pi^2}(\mathrm{L}i_3(r) - \mathrm{L}i_3(-r)).$$

Letting $a = b = 1$ we recover Smyth's well-known result that

$$m(x + y + z + 1) = \frac{7}{2\pi^2}\zeta(3).$$

Acknowledgements
The author is grateful to Fernando Rodriguez Villegas for proposing this avenue of exploration, as well as for his support and further ideas during the writing process. Thanks also to Matilde Lalin for proofreading the text and suggesting improvements.

References

[Bl] S. Bloch, Function theory of polylogarithms, *Structural properties of polylogarithms: L. Math. Surveys Monogr.* **37** (1991) 275–285.

[Bo1] D.W. Boyd, Mahler's measure and special values of L-functions, *Exper. Math.* **7** (1998), 37–82.

[Bo2] D.W. Boyd, Mahler's measure and invariants of hyperbolic manifolds, in: B.C. Berndt et al, eds., *Number Theory for the Millennium: Proc. Millennial Conf. Number Theory* (A. K. Peters, Boston, 2002) 127–143.

[BRV] D.W. Boyd & F. Rodriguez Villegas, Mahler's measure and the dilogarithm (I), *Canadian Journal of Mathematics* **54** (2002) 468–492.

[La] S. Lang, *Introduction to Modular Forms* (New York, NY, 1976)

[Ma] V. Maillot, Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables, *Mém. Soc. Math. Fr. (N.S.)* **80** (2000), 129pp.

[RV] F. Rodriguez Villegas, Modular Mahler measures. I, *Topics in number theory* (University Park, PA, 1997), 17–48, Math. Appl., 467, Kluwer Acad. Publ., Dordrecht, 1999.

[Sm] C.J. Smyth, An explicit formula for the Mahler measure of a family of 3-variable polynomials, to appear in *J. Theorie Nombres Bordeaux*.

[Za] D. Zagier, The remarkable dilogarithm, *Jour. Math. Phys. Sci.* **22** (1988) no. 1, 131–145.