Making a Mathematical Model for a Square Pendulum
Fall 2017

Introduction

It often happens in science that we know how to describe a phenomenon with a mathematical equation before we know the theory behind it. One of the best examples is the orbit of the planets. Kepler was able to deduce three laws of planetary motion directly from experimental observations. These laws were merely descriptive; they gave certain features of how the planets moved, but made no pretense at explaining why. It was only later that Newton showed how these laws could be derived from two fundamental ideas: his second law of motion and his inverse square law of gravity.

Today you will be following in Kepler’s footsteps. You have a collection of square pendulums of different sizes that can be suspended from one corner. The pendulums oscillate each with a certain period (the time for one complete back-and-forth oscillation). Your goal is to discover an equation that relates the period, $T$, to the length, $L$, of the side of the square.

Experimental Details

The experiment itself is straightforward. The period is measured with a photogate set to “PEND” (pendulum) mode, memory “on”. Position the photogate so that the point of the pendulum when at rest interrupts the beam. Let the pendulum swing, and repeatedly hit the “Reset” button to collect your data (don’t stop and restart the pendulum!). Be careful, however, to use only small amplitude oscillations – the pendulum point should swing just enough to break the photogate beam. You can raise or lower the photogate on the timer or the pendulum support bar to accommodate different lengths. Use a meter stick with caliper jaws to measure the length of the side. Be careful not to bend the small wire support on the corners; when you place the pendulum on the bench, do it so that the support wire hangs over the end of the table.

The idea now is to use the laboratory and report writing skills you have learned, including the following:

- Drawing and labeling a picture of the apparatus.
- Setting up a data table. You’ll find it easier in this experiment to keep your measurements in centimeters.
- The correct procedure for collecting measurements with the implement provided (e.g. both meter stick caliper measurements must be recorded in your data table).
- The rules of data taking and graph creation to plot $T$ vs. $L$. You don’t know what the graph will look like before you start the experiment, but you should expect to find a smooth function.
- Using KaleidaGraph to try out various types of fits and make a final graph.
- Explaining the purpose of your measurements and calculations.
- The proper presentation of your results. Be sure to use $T$ and $L$ (not $x$ and $y$) when presenting your model equation.
- Writing a thoughtful discussion.

Hints on Finding the Best Fit

Two ideas will help you here. One is that there should be no systematic pattern in the residuals, i.e. the data should be scattered about both sides of the best fit in a more or less random fashion. (Of course, if you only have a few data points, this strategy is impossible to implement!). The second is that if you extend the fit line backward to $L = 0$ it should come pretty close to $T = 0$. If something weird (un-physical) is predicted by the best fit as $L$ approaches 0 then you can bet you have chosen the wrong function to fit your data.
KaleidaGraph Notes

1. Enter your data and create a $T$ vs. $L$ scatter plot of your data; don’t add {0, 0} as a data point this week. You will need to change the axis scale so that the origin is shown (Select the graph, then choose Plot \Rightarrow Axis Options; set the Minimum value to 0 for the x- and y-axes).

2. Select one of the built-in functions from the Curve Fit menu (Linear, Polynomial (2nd order), Exponential, Logarithmic or Power), and determine which function best fits your data. Don’t forget to deselect a fit before choosing another.

3. With the model of the best-fitting function chosen, you can then create your own “user-defined” fit, so that the uncertainties in the parameters will be calculated.
   a. Choose Curve Fit \Rightarrow General \Rightarrow New Fit. Note that this will keep the original fit on your graph, which will make it easier to ensure that you have created the user-defined fit correctly.
   b. Click the Define… button, and replace the contents with your equation. For example, a linear fit would be entered as follows:
      \[ a \times x + b; \ a = 1; \ b = 1 \]
      The statement “a = 1; b = 1” initializes ‘a’ and ‘b’ at unity, giving KaleidaGraph a place to begin its calculations. Use 1 for your custom fit; don’t use the values of ‘a’ and ‘b’ that were calculated using the generic function from step 3!
   c. Once you have the function defined, deselect the fit from step 3 above so that only the user-defined fit remains on your graph.

4. Your best fit should extend across the width of the graph. Print your graph, but don’t close it yet; you’ll be making another plot in a few minutes.

5. Record your model equation in your report; include the parameters and their uncertainties.

Testing Your Model

Any law of nature is useful only if it has predictive power. If you have found a good model for the square pendulum, then you should be able to predict the period of any square pendulum provided you know the side $L$.

You will be given one pendulum that you didn’t use before and asked to predict its period. Your prediction should be of the form $T = \_\_\_\_ \pm \_\_\_\_ sec$. Otherwise the chances are nearly zero that you will get the exact value.

How do you estimate the uncertainty in the predicted $T$? You could look at how repeatable any given period measurement is. However, this ignores uncertainties in the length and slight variations in pendulum construction. The answer is hidden in our complete data, which lie above and below the best-fit line by various amounts (the residuals). Therefore, a good estimate is to find the “typical” residual, which will take into account all of the experimental uncertainties. The best estimate of the “typical” residual is to calculate the Root Mean Square Residual:

\[
\text{RMS Residual} = \sqrt{\frac{\sum_{i=1}^{N} R_i^2}{N}} \quad \text{(Eqn.1)}
\]

where $N$ is the number of data points, and $R$ is the residual for each point (Note: this is different than the value of $R$ calculated by KaleidaGraph. Ignore KaleidaGraph’s value of $R$)! You will find the square root of the average of the squares of the residuals using the procedure outlined on the next page.
The quantity in the numerator of Eqn. 1 is the sum of the squares of the residuals, SSR (also known as Chi Square, \( \chi^2 \)):

\[ SSR = \sum_{i=1}^{N} R_i^2 \]  

(Eqn.2)

Therefore, we can rewrite Eqn. 1 as:

\[ \text{RMS Residual} = \sqrt{\frac{SSR}{N}} \]  

(Eqn.3)

(Recall that the SSR will appear on your graph (the \( \text{Chisq} \) value) when you create a “user-defined” fit.) The RMS Residual value indicates the typical uncertainty in a single measurement.

1. Use Eqn. 3 to calculate the RMS residual from your KaleidaGraph results (note that “1e–5” is computer-speak for “1\times10^{-5}”). Your estimate of the uncertainty in \( T \) for your prediction will be twice the RMS Residual value you calculate.

2. Once you have your prediction, explicitly write out the range of possible periods for the unknown pendulum. Perform measurements (which should be recorded in a table) of the unknown pendulum’s period and calculate the average period.

Is There a Systematic Error in Your Data?

Create a new graph of your residuals vs. \( L \), and think about its meaning. You can use the residuals to look for a systematic error that might occur during the experiment; a pattern in this graph might indicate such an error. You might wish to increase the vertical scale to “zoom out” from the graph a little bit (Set the vertical scale to range between \( \leq 0.01 \) units). Print a copy of this graph for your report.

Discussion

- Summarize your results by presenting the following:
  - Your model equation. You should also discuss the reasons you rejected your next best choice for the model equation.
  - The parameters, their uncertainties and the SSR value (with appropriate units!)
  - The predicted period and its uncertainty.
  - The success of your model.
- Discuss the possible sources of error.
- Include a comment on the appearance of your residuals plot. Did you notice any pattern in the residuals?