1 C^*-algebras

I primarily study *operator algebras*, a sub-field of functional analysis with rich algebraic and topological aspects. Most of my research has revolved around studying C^*-algebras attached to groupoids and directed graphs. A C^*-algebra is a (complex) Banach algebra A with an antilinear involution $* : A \to A$ such that $(ab)^* = b^*a^*$ and $||a^*a|| = ||a||^2$ for all $a, b \in A$. Some basic examples are:

- (Matrix algebras) If $n \geq 1$, then $M_n(\mathbb{C})$, the set of $n \times n$ complex matrices forms a C^*-algebra when equipped with operator norm, standard algebraic operations on $M_n(\mathbb{C})$, and the Hermitian adjoint operation as $*$-operation.
- (Bounded operators) If $H = \ell^2(\mathbb{N})$ is a countably-infinite dimensional Hilbert space, then the set of all bounded linear operators on H forms a C^*-algebra denoted $B(H)$, with operations as in the previous example.
- (Compact operators) a linear operator $T : H \to H$ is *compact* if the image of the closed unit ball under T has compact closure in H. The set of compact operators, denoted $\mathcal{K}$, forms a C^*-subalgebra of $B(H)$.
- (Abelian C^*-algebras) If X is a compact Hausdorff space, then $A = C(X)$, the set of continuous functions on X equipped with supremum norm, pointwise operations, and pointwise conjugation as $*$-operation, forms a C^*-algebra.

Arising initially in quantum mechanics, C^*-algebras have found numerous applications in physics and areas of mathematics such as group theory, geometry, and topology.

2 Groupoids

A *groupoid* consists of a set G along with a set $G^{(2)} \subset G \times G$ of *composable pairs*, a multiplication operation $\circ : G^{(2)} \to G$, written $(\alpha, \beta) \mapsto \alpha\beta$, that is associative: $\alpha(\beta\gamma) = (\alpha\beta)\gamma$, and an involutive inverse operation $\alpha \mapsto \alpha^{-1}$ that is cancellative: $(\alpha\beta)\beta^{-1} = \alpha = \gamma^{-1}(\gamma\beta)$.

Here are a some key examples of groupoids:

- Suppose that H is a discrete group that acts via homeomorphisms on a locally compact Hausdorff space X, written $h.x = \alpha_h(x)$. Then the Cartesian product $G = H \times X$ forms a groupoid called the transformation groupoid, with multiplication operation
$$(h_1, y)(h_2, x) = (h_1h_2, x) \text{ if } y = h_2.x.$$
(This implicitly defines the inverse map as well as the set of composable pairs.) Setting $X = \{*\}$ we obtain every discrete group H as a groupoid.
- Suppose that $E = (E^0, E^1, r, s)$ is a directed graph, where E^0 is the set of vertices, E^1 is the set of edges, and $r, s : E^1 \to E^0$ are the range and source maps. The subset of $(E^1)^{\mathbb{N}}$ consisting of all sequences $x = (e_n)_{n=1}^{\infty}$ so that $s(e_n) = r(e_{n+1})$ for every $n \geq 1$ is called the *infinite path space* is denoted E^{∞}. The shift $\sigma : E^{\infty} \to E^{\infty}$ deletes the first edge from a sequence; two sequences $x, y \in E^{\infty}$ are equivalent with lag $n \in \mathbb{Z}$, written $x \sim_n y$, if there exist $j, k \geq 0$ so that $\sigma^j x = \sigma^k y$ and $j - k = n$. The *path groupoid* G_E is defined to be $\{(x, n, y) : x \sim_n y\} \subset E^{\infty} \times \mathbb{Z} \times E^{\infty}$. The multiplication is $(x, n, y)(y, m, z) = (x, n + m, z)$.

3 Groupoid C^*-algebras

The upshot of groupoids is that they generalize groups and provide a very rich source of C^*-algebras. In the study of groupoid C^*-algebras we try to tie properties of a groupoid G to interesting properties of its associated C^*-algebras.

In order to define a C^*-algebra from a groupoid, some terminology has to be introduced:

- The *unit space* of a groupoid G consists of all elements u so that $uu = u$. Unlike in groups, there can be many unit elements. The unit space is denoted by $G^{(0)}$.
- The *source and range maps* $r, s : G \to G$ map are defined by $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$. Each of these maps G onto the unit space.
- A *topological groupoid* is a groupoid equipped with a topology so that the multiplication and inverse maps become continuous. In addition, we require our topologies to be second-countable, locally compact, and Hausdorff (although the first and last conditions are sometimes omitted).
- An *étale groupoid* is a topological groupoid so that the range and source maps $r, s : G \to G$ are local homeomorphisms. This is equivalent to the existence of a topology base consisting of *bisections*, that is, sets B so that $r|_B$ and $s|_B$ are both homeomorphisms onto their range.

With all that defined, we can start building C^*-algebras. Begin with a locally compact, second countable, Hausdorff étale groupoid G. First take the vector space $C_c(G)$ of all continuous functions $f : G \to \mathbb{C}$ that have compact support. This becomes a $*$-algebra by defining a convolution product $f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ and an involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$. The hard bit comes in defining the norm: there are at least two natural candidates: $||\cdot||$ (full norm) and $||\cdot||_r$ (reduced norm). Taking the completion of $C_c(G)$ with respect to these norms yields two C^*-algebras: $C^*(G)$ and $C^*_r(G)$. I've mostly studied the latter C^*-algebra, which has the advantage of being given by a specific representation on Hilbert space. For more details, please see [13].

Each example groupoid given in the previous section gives rise to a C^*-algebra. The C^*-algebras associated to $G = H \times X$ are typically written $C(X) \rtimes H$ and $C(X) \rtimes_r H$, called the *full* and *reduced crossed products*. If E is a directed graph then the (unique) C^*-algebra associated to $G = G_E$ is denoted $C^*(E)$, called the *graph C^*-algebra*. The graph C^*-algebra is generated by projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{s_e\}_{e\in E^1}$ satisfying equations called the *Cuntz-Krieger relations*, see [11].

A prototypical task in this area of study is to determine how the structure of the directed graph E affects the C^*-algebra $C^*(E)$. For example, a directed graph E has no directed cycles if and only if the corresponding C^*-algebra $C^*(E)$ can be written as an inductive limit of direct sums of matrix algebras (that is, if $C^*(E)$ is an *AF algebra*). Many other important operator algebraic properties of $C^*(E)$ can be "read" from the parent graph E–hence the nickname of graph algebras as "C^*-algebras we can see." In Section 4, Section 5, and Section 6 there are results obtained by myself (some in joint work with Gabriel Nagy) along these lines.

4 Continuous-trace graph and k-graph C^*-algebras

- A $*$-representation of a C^*-algebra A on a Hilbert space H is a $*$-algebra morphism $\pi : A \to B(H)$. Two such representations $\pi_i : A \to B(H_i)$, $i = 1, 2$, are called *unitarily equivalent* if there is a unitary operator $U : H_1 \to H_2$ so that $U\pi_1(a) = \pi_2(a)U$ for all $a \in A$. The set of equivalence classes of representations is called the *spectrum* of A, denoted $\widehat{A}$. This can be equipped with a locally compact topology called the Jacobson topology, see [12]. As a simple example, if $A = C(X)$, then the spectrum of A is homeomorphic to X.

- A C^*-algebra A is said to be *continuous-trace* if $\widehat{A}$ is a Hausdorff space and certain projection operators can always be found "locally" (for a more precise definition, see [12]). Being continuous-trace is a difficult condition to satisfy, but it carries a wealth of useful structural information.

- If E is a directed graph, a *directed cycle* is a finite sequence of edges $\lambda = e_1 \dots e_n$ so that $s(e_k) = r(e_{k+1})$ for $k = 1, \dots, n-1$ and $s(e_n) = r(e_1)$. An *entrance* to a cycle $e_1 \dots e_n$ is an edge f so that $r(f) = r(e_k)$ but $f \neq e_k$ for some $k = 1, \dots, n$.

- An *ancestry pair* for vertices v, w is a pair of directed paths (λ, μ), neither containing a cycle, such that $r(\lambda) = v$ and $r(\mu) = w$ and $s(\lambda) = s(\mu)$. One ancestry pair can contain another as $(\lambda\nu, \mu\nu) \supset (\lambda, \mu)$, if ν is a finite path with $r(\nu) = s(\lambda)$. The *minimal ancestry pairs* contain no other ancestry pairs.

In [2] I showed the following:

Theorem 4.1 ([2, Thm 4.7])**.** *Let E be a directed graph. Then $C^*(E)$ has continuous trace if and only if both of the following conditions hold:*

1. *no cycle of E has an entrance;*
2. *any pair of vertices (v, w) has at most finitely many minimal ancestry pairs (α, β).*

A *k-graph* (or *higher-rank graph*) is a countable category Λ equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ (where the range is a category under addition with a single object) that satisfies the *factorization property*: if $d(\lambda) = m + n$ for $m, n \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda$ satisfying $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. (For details, please see [7] or [11, Ch. 10].) The morphisms are called *paths*; the paths of degree 0 are called *vertices* and correspond exactly with the objects of the category. Thus path has a range vertex and source vertex owing to the structure of the category. One can produce a groupoid G_Λ from a k-graph by using suitably defined "infinite paths" within Λ; these are useful for studying C^*-algebras of k-graphs.

Each (suitably well-behaved) k-graph Λ gives rise to a C^*-algebra $C^*(\Lambda)$. The terms "principal" and "row-finite" are technical requirements; for details see [2].

Theorem 4.2 ([2, Thm. 5.15])**.** *Let Λ be a principal row-finite k-graph with no sources. Then $C^*(\Lambda)$ has continuous trace if and only if any two vertices of Λ have finitely many minimal ancestry pairs.*

5 Stability

- A C^*-algebra is called *stable* if $A \otimes \mathcal{K} \cong A$, where $\mathcal{K}$ denotes the algebra of compact operators on the Hilbert space $H = \ell^2(\mathbb{N})$. Here $A \otimes \mathcal{K} \cong \overline{\bigcup M_n(A)}$, where $M_n(A) \hookrightarrow M_{n+1}(A)$ embeds in the upper left corner.
- A tracial state on a C^*-algebra A is a positive linear function $\phi : A \to \mathbb{C}$ of norm 1 such that $\phi(ab) = \phi(ba)$ for all $a, b \in A$. One cannot define a tracial state on a stable C^*-algebra.
- A *graph trace* on a directed graph E consists of an ℓ^1-function $f : E^0 \to [0, \infty)$ satisfying Cuntz-Krieger equations (see [14, Def. 2.2]).
- Two projections p and q in a C^*-algebra A are *equivalent* if there exists $s \in A$ so that $s^*s = p$ and $ss^* = q$, we write $p \sim q$ in this case. If p is similar to q and $q \leq r$, then we write $p \lesssim r$.
- A directed cycle $\lambda = e_1 \dots e_n$ in a graph E is *left-finite* if one can only access a finite number of vertices by following directed paths which begin at the vertex $r(e_1)$.

The following is a refinement of the work in [14], which had a small gap in its proof which I repaired.

Theorem 5.1 ([1, Thm. 3.31]). *Let E be a directed graph. The following are equivalent:*

(1) $C^*(E)$ is stable;

(2) $C^*(E)$ has no nonzero unital quotients and no tracial states;

(3) E has no left finite cycles and no bounded graph traces;

(4) E has no left finite cycles, no left finite singular vertices, and no bounded graph traces;

(5) for any vertex $v \in E^0$ and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$;

(6) for any vertex $V \subset E^0$ and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $\sum_{v \in V} p_v \lesssim \sum_{w \in W} p_w$.

I was able to generalize parts of this theorem to the context of k-graph algebras, where the notion of graph trace is replaced by k-graph trace. A characterization of the k-graphs with stable C^*-algebras seems out of reach, but I was able to obtain the following necessary condition:

Theorem 5.2 ([1, Cor. 7]). *If Λ is a row-finite k-graph with no sources so that $C^*(\Lambda)$ is stable, then Λ has no k-graph traces.*

I also found a sufficient condition for a k-graph to satisfy to produce a stable C^*-algebra, very much in line with the ideas from [14].

Theorem 5.3 ([1, Thm. 6]). *Let Λ be a row-finite k-graph with no sources. Suppose that every vertex $v \in \Lambda^0$ is left infinite. Then $C^*(\Lambda)$ is stable.*

I also found extensions of this work to the context of groupoids, which can be found in [1, Sec. 5]. I plan to extend it to the context of inverse semigroup C^*-algebras, which can be described using a groupoid model.

6 Tracial states and inclusions

Initially inspired by the proof of Theorem 5.1, Gabriel Nagy and I invented a flexible method for defining tracial states on $C*$-algebras in [4].

- An *inclusion* of C^*-algebras is just $B \subset A$ where B and A are both C^*-algebra. Typically we only look at *non-degenerate* inclusions, where B carries an approximate identity (see [8]) for A.
- A *conditional expectation* for $B \subset A$ consists of a completely positive map $P : A \to B$ so that $P(b) = b$ for all $b \in B$. Such maps are very useful in operator algebras. These arise naturally in the groupoid context by sending $C_c(G) \to C_c(G^{(0)})$ via restriction and extending to a map $P : C_r^*(G) \to C_0(G^{(0)})$.
- A *normalizer* for a subalgebra $B \subset A$ is an element $n \in A$ so that nbn^* and n^*bn belong to B for every $b \in B$. The inclusion is *regular* if $N(B)$ spans a dense subset of A. If the inclusion is non-degenerate then nn^* and n^*n belong to B for all $n \in N(B)$.

A state on B is *fully invariant* if $\phi(nbn^*) = \phi(n^*nb)$ for all $b \in B$ and $n \in N(B)$. It is trivial to see that if ϕ is a tracial state on A, then $\phi|_B$ is a fully invariant state on B. Our work in [4] focused on determining if every fully invariant state on B arises in this fashion; in other words, does every fully invariant state ϕ on $B \subset A$ induce a tracial state $\tilde{\phi}$ on A such that $\tilde{\phi}_| B = \phi$?

Theorem 6.1 ([4]). *Let $B \subset A$ be an abelian C^*-subalgebra which is the range of a conditional expectation $P : A \to B$ such that $P(nan^*) = nP(a)n^*$ for all $n \in N(B)$ and $a \in A$. Then for any fully-invariant state ϕ on B, the composition $\phi \circ P$ is a tracial state on A.*

This analysis has a natural interpretation in the context of étale groupoids. A Radon probability measure μ on the unit space of an étale groupoid is called *totally balanced* if for any Borel $X \subset G^{(0)}$ and any open bisection $B \subset G$, we have $\mu(BXB^{-1}) = \mu(B^{-1}BX)$. (This is the groupoid analogue of a fully invariant state on $C_0(G^{(0)})$.) Étale groupoid C^*-algebras always have a conditional expectation $P : C_r^*(G) \to C_0(G^{(0)})$, given by extending the restriction map $C_c(G) \mapsto C_c(G^{(0)})$.

Theorem 6.2 ([4]). *Let G be a second countable locally compact Hausdorff étale groupoid.*

- *Any totally balanced measure μ on $G^{(0)}$ induces a tracial state on $C_r^*(G)$ via $\tau_\mu(f) = \int P(f)d\mu$.*
- *If G is* principal*, in the sense that $r(\gamma) = s(\gamma)$ implies $\gamma \in G^{(0)}$, then every tracial state on $C_r^*(G)$ arises in this fashion.*

7 C^*-simplicity for étale groupoid C^*-algebras

Each discrete group is an étale groupoid and in particular gives rise to a reduced group C^*-algebra. A discrete group Γ is called *C^*-simple* if its reduced group C^*-algebra is simple in the sense of having no closed two-sided ideals. Recently the study of C^*-simple groups has been revitalized by the work begun in [6], where the behavior of a certain dynamical system associated to G was shown to completely answer the question of C^*-simplicity. Gabriel Nagy and I sought to find some extension of this work to the realm of groupoids, and we proved the following theorem. The assumption is that G is a locally compact, second countable, étale, Hausdorff groupoid (a fairly standard collection of hypotheses to impose). The set $G^{(0)}_{\text{r-cont}}$ is a certain subset of $G^{(0)}$, whose (somewhat technical) definition appears in Sec. 7 of [5].

Theorem 7.1 ([5, Thm. 7.6])**.** *Assume that there is some unit $u_0 \in G^{(0)}_{\text{r-cont}}$ for which the (discrete countable) group $\mathrm{IntIso}(G)_{u_0}$ is C^*-simple. Then the following are equivalent:*

(i) G is minimal (that is, no nontrivial closed subset $X \subset \mathcal{G}$ satisfies $r(s^{-1}(X)) = X$);

(ii) $C^*_r(G)$ is simple.

As in Section 2, whenever a discrete group G acts upon a locally compact Hausdorff space Q via homeomorphisms, we can define a related étale groupoid $Q \rtimes G$. This is called the *transformation groupoid* of the dynamical system. For a point $q \in Q$ we define G°_q to be the group of all elements of G that fix some neighborhood of q

Theorem 7.2 ([5, Thm. 8.7])**.** *If there is a point $q_0 \in Q$ such that $G^\circ_{q_0}$ is C^*-simple, then the following conditions are equivalent:*

(i) the action of G on Q is minimal (no non-trivial closed subset of Q satisfies $gQ = Q$ for all g);

(ii) the reduced C^*-algebra $C^*_r(Q \rtimes G)$ is simple.

This result is an improvement over an existing result of Ozawa which required that there exist a point $q \in Q$ such that the (larger) group G_q is C^*-simple. We hope to sharpen our results to apply to wider classes of groupoid C^*-algebras.

8 Combinatorics and graph theory

Some of my recent work has taken me out of the realm of functional analysis and into pure combinatorics and graph theory.

- The classical *cops and robber game* is played by two players on an undirected graph. The cop player is allowed to place a certain number of cop tokens on vertices of the graph; on the next turn the robber player can place the robber token on any unoccupied vertex of the graph. Then the players take turns: each token can be moved from its vertex to any adjacent vertex. The cop player wins if they place a cop token on the same vertex as the robber token (this is called *capture*); the robber player wins if this never happens. If G is the graph, the *cop number* of G, denoted $c(G)$, is the minimum number of cops that must be placed initially in order to ensure capture. Typical theorems relate structural features of G (such as retracts or graph product structure) to the cop number.

- A variation of cops and robber, called *containment*, was invented by Natasha Komarov and John Mackey (introduced in [3], although they described it before I joined the project). The setup is similar, but the cops are placed on *edges* instead of vertices; the cop player wins if the robber token occupies a vertex all of whose incident edges carry cops (this is called *containment*). The *containment number* of G, denoted $\xi(G)$, is the minimum number of cops that must be placed initially to ensure containment.

A conjecture in [3] is to prove that $\xi(G) \leq c(G) \cdot \Delta(G)$, where $\Delta(G)$ is the maximum vertex degree of the graph G. It is shown in [9] that random graphs asymptotically satisfy this bound.

My contribution to [3] was to prove a related result.

- A graph homomorphism between undirected graphs G and H is just a map $V(G) \to V(H)$ between the vertex sets that preserves adjacency.
- A subgraph $H \subset G$ is called a *retract* if there is a graph homomorphism $\phi : G \to H$ that fixes every vertex of H.
- If $H \subset G$ is a subgraph and $\phi : G \to H$ is a retract, we call ϕ *cubical* if $v \in V(G) \setminus V(H)$ being adjacent to $h \in V(H)$ implies $h = \phi(v)$.
- If $H \subset G$ is a subgraph, then for each $v \in V(H)$ define the *degree discrepancy* of to be $dd(G, H, v) := \deg_G(v) - \deg_H(v)$. The *degree discrepancy of* H is

$$dd(G, H) := \max_{v \in V(H)} dd(G, H, v).$$

Theorem 8.1 ([3, Thm. 2.3])**.** *If the subgraph $H \subset G$ is a retract of G, then $\xi(H) \leq \xi(G)$*

In the following result, $\xi(G - H)$ refers to the maximum containment number of a connected component of the graph $G - H$ obtained by deleting H from G.

Theorem 8.2 ([3, Thm. 2.7])**.** *Let $H \subset G$ be a cubical retract of G. Then*

$$\xi(G) \leq \max\{\xi(H), \xi(G - H)\} + dd(G, H) + \Delta(H) - 1$$

9 Plans for future research

One of my future projects is to extend the work of David Pitts and Vrej Zarikian on largeness of inclusions (see [10], [16], [15]) to the realm of étale groupoids. Their existing work highlights very interesting connections between algebraic properties of an inclusion $B \subset A$ and the original topological data used to specify A as a commutative or crossed-product C^*-algebra. Their ideas seem to mesh well with the groupoid approach to C^*-algebras. So far I have obtained a few preliminary results connecting graph and groupoid C^*-algebras to their ideas, see citeCRYTSERJMM. I also plan to sharpen and extend the results on simplicity of groupoid C^*-algebras that I've obtained with Gabriel Nagy.

Outside of the realm of C^*-algebras, I'm planning on learning more about the theory of pursuit games on directed graphs. These problems have a completely different, algorithmic flavor to them, very different from the functional analytic problems that I'm used to. I believe that they will provide a good source of undergraduate math projects, please see the next section.

10 Research opportunities with undergraduates

The groupoid model of graph C^*-algebras offers great opportunities for undergraduate research. The graph and graph groupoid are both so simple and easy to understand that undergraduates can then set out to discover graph theoretic properties which guarantee the needed groupoid behavior. Whatever results obtained then "trickle up" to become theorems about the corresponding C^*-algebras.

In 2016, I mentored a summer REU at Kansas State University. I avoided giving them lots of functional analysis material to learn by translating problems in C^*-algebras to problems in graph theory. The three students in my group proved some interesting results regarding extreme points in the set of graph traces (which correspond to tracial states), and learned some interesting theorems about convex geometry along the way. For example, I had them learn Carathéodory's theorem for convex sets, and we discussed its possible relevance to the problems. In general I was impressed at their ability to formulate new ideas, as well as the speed with which they picked up LaTeX.

At St Lawrence University I have supervised two undergraduate research projects. The first was an Honors Thesis project on measure theory, where a student of mine wrote up an expository document on stochastic processes and the proof of the Kolmogorov Extension Theorem. Finding a math research topic that suited a student headed for a Stats PhD program took a little preparation but it was successful. The second project that I undertook was a McNair Scholar project about the Banach-Tarski paradox. The student who completed the project had scant background (no abstract algebra or linear algebra) but she learned all that was necessary and wrote a very good paper. We are currently refining her work into a more complete paper.

My recent work in combinatorics and graph theory could prove a great source of undergraduate research problems. Students can easily spend time coming up with examples of graphs possessing certain properties, or learning the code needed to check large numbers of examples. The proofs of theorems are also more accessible in this case. Many times the essence of a proof in a pursuit game comes in describing an explicit algorithm for either capturing a robber or evading the cops, either of which can be grasped by undergraduates with minimal background.

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